1.1 The Language of Mathematics
Expressions versus Sentences

a hypothetical situation

Imagine the following scenario: you are sitting in class, and the instructor passes a small piece of paper to each student. You are told that the paper contains a paragraph on *Study Strategies for Students of Mathematics*; your job is to read it and paraphrase it. Upon glancing at the paper, however, you observe that it is written in a foreign language that you do not understand!

the importance of language

Now, is the instructor being fair? Of course not. Indeed, the instructor is probably trying to make a point. Although the *ideas* in the paragraph may be simple, there is no access to these ideas without a knowledge of the *language* in which the ideas are expressed. This situation has a very strong analogy in undergraduate mathematics courses. Students frequently have trouble understanding the ideas being presented; not because the ideas are difficult, but because they are being presented in a foreign language—the language of mathematics.

Study Strategies for Students of Mathematics

A list of *Study Strategies for Students of Mathematics* (in English) appears in this text after the *Preface*. Be sure to read both of these sections.

characteristics of the language of mathematics

The language of mathematics makes it easy to express the kinds of thoughts that mathematicians like to express. It is:

- precise (able to make very fine distinctions);
- concise (able to say things briefly);
- powerful (able to express complex thoughts with relative ease).

The language of mathematics can be learned. However, it requires the efforts needed to learn any foreign language.

a major goal of this text

Throughout this text, attention is paid not only to the ideas presented, but also to the language in which these ideas are expressed. Besides understanding calculus, a major goal of this course is for you to improve your skills in *reading* and *writing* mathematics. These skills can be carried with you into any setting where mathematics is used to express ideas.

complete and correct mathematical sentences

You can't learn to read without reading. *So read the text.* You can't learn to write without writing. So you will be given ample opportunities to practice writing complete and correct mathematical sentences.

We now begin our study of the language of mathematics. The ideas introduced here will be elaborated on throughout the text.

English nouns versus mathematical 'nouns'

In English, a *noun* is a word that names something. An English noun is usually a person, place, or thing; for example, *Julia*, *Idaho*, and *rat*. Note that there are conventions regarding nouns in English; for example, proper names are capitalized.

The mathematical analogue of a noun is called an *expression*.

mathematical expression

A *mathematical expression* is a name given to some mathematical object of interest. The phrase '*mathematical expression*' is usually shortened to '*expression*'.

conventions regarding the naming of mathematical 'nouns'

In mathematics, an 'object of interest' is often a number, a set, or a function. There are conventions regarding the naming of 'nouns' in mathematics, just as there are in English. For example, real numbers are usually named with lowercase letters (like a, x, t, α, β, and γ), whereas sets are usually named with capital letters (like A, B, and C). Such conventions are addressed throughout the text.

Without sets and functions, modern mathematics could not exist. Sets, and the important sets of numbers, are reviewed throughout Chapter 1. Functions are discussed in Chapter 2. The Algebra Review at the end of the current section reviews the most commonly used Greek letters, and the real numbers.

By themselves, nouns are not extremely useful. It is when nouns are used in *sentences* to express complete thoughts that things get really interesting.

sentences

A declarative English *sentence* begins with a capital letter, ends with a period, and expresses a complete thought:

Many students are a bit apprehensive of their first Calculus course.

A *mathematical sentence* must also express a complete thought. However, there are a lot of symbols (and layouts) available in the construction of mathematical sentences that are not available in the construction of English sentences.

Many students have trouble distinguishing between mathematical *expressions* and mathematical *sentences*. Exercises and examples that help you understand the difference will appear throughout the text.

how to decide if something is a sentence

A good way to decide if something is a *sentence* is to *read it out loud*, and ask yourself the question: Does it express a complete thought? If the answer is 'yes', it's a sentence.

The difference between expressions and sentences is explored in the next example.

EXAMPLE

sentences versus expressions

Problem: Classify the entries in the list below as:

- an English noun
- a mathematical expression
- a sentence

In any sentence, circle the verb. Try to fill in the blanks yourself before looking at the solutions.

(For the moment, don't worry about the *truth* of sentences. This issue is addressed in the next example.)

1.	cat	________
2.	x	________
3.	The word 'cat' begins with the letter 'k'.	________
4.	$1+2=4$	________
5.	$(a+b)^2$	________
6.	$2x-1=0$	________
7.	The cat is black.	________
8.	$(a+b)^2=a^2+2ab+b^2$	________
9.	$-3t<2$	________
10.	$y+y+y$	________
11.	$y+y+y=3y$	________
12.	$(a+b)^2=a^2+b^2$	________
13.	This sentence is false.	________
14.	$x^2<0$	________
15.	$1+\sqrt{2}$	________

Solution:

1.	cat	English noun
2.	x	mathematical expression
3.	The word 'cat' begins with the letter 'k'.	sentence
4.	$1+2=4$	sentence
5.	$(a+b)^2$	mathematical expression
6.	$2x-1=0$	sentence
7.	The cat is black.	sentence
8.	$(a+b)^2=a^2+2ab+b^2$	sentence
9.	$-3t<2$	sentence
10.	$y+y+y$	mathematical expression
11.	$y+y+y=3y$	sentence
12.	$(a+b)^2=a^2+b^2$	sentence
13.	This sentence is false.	sentence
14.	$x^2<0$	sentence
15.	$1+\sqrt{2}$	mathematical expression

Note that sentences express a complete thought, but nouns (expressions) do not. For example, read aloud: x. *What about* x? Now read aloud: $2x-1=0$. Here, a complete thought about object 'x' has been expressed.

EXAMPLE
truth of sentences

Problem: Consider the entries in the previous example that are *sentences*. Which are true? False? Are there possibilities other than true and false?

Solution:

3.	The word 'cat' begins with the letter 'k'.	FALSE
4.	$1 + 2 = 4$	FALSE
6.	$2x - 1 = 0$	The truth of this sentence (true or false) depends on the choice of x. If x is $1/2$, then it is true. Otherwise, it is false. Sentences such as these are studied in more detail in the next section.
7.	The cat is black.	The truth of this sentence cannot be determined out of context. If the cat being referred to is indeed black, then the sentence is true. Otherwise, it is false.
8.	$(a + b)^2 = a^2 + 2ab + b^2$	Here, it is assumed that a and b represent numbers. Then, this sentence is (always) true: its truth does *not* depend on the numbers chosen for a and b. ♣ Why?
9.	$-3t < 2$	It is assumed that t represents a number. This sentence is sometimes true, sometimes false, depending on the number chosen for t. In sentences such as these, mathematicians are often interested in finding the choices that make the sentence TRUE.
11.	$y + y + y = 3y$	TRUE, for all real numbers y.
12.	$(a + b)^2 = a^2 + b^2$	The truth of this sentence depends on the choices for a and b. For example, if $a = 0$ and $b = 1$, then it is true (♣ check). If $a = 1$ and $b = 1$, then it is false (♣ check).
13.	This sentence is false.	IF this sentence is true, then it would have to be false. IF this sentence is false, then it would have to be true. So this sentence is not true, not false, and not sometimes true/sometimes false.
14.	$x^2 < 0$	It is assumed that x represents a real number. Since every real number, when squared, is nonnegative, this sentence is (always) false.

EXERCISE 1

♣ Write a few (English) sentences that discuss the difference between mathematical *expressions* and *sentences*.

(Remember that the ♣ (clubsuit) symbol means that student input is required.)

EXERCISE 2

♣ 1. In algebra, how did you go about finding the choice(s) for x that make the equation $2x - 1 = 0$ true? Do it.

♣ 2. In algebra, how did you go about finding the choice(s) for t that make the inequality $-3t < 2$ true? Do it.

♣ 3. What happens if you take the usual algebra approach, and try to 'solve' the equation $y + y + y = 3y$?

♣ 4. What are all the possible choice(s) for a and b that make the equation $(a+b)^2 = a^2 + b^2$ true? Be sure to write complete sentences in your answer.

♣ 5. What name is commonly given to English sentences that are *intentionally* false? To English sentences that are *nonintentionally* false?

EXERCISE 3

sentences versus expressions

♣ Classify each entry in the list below as: an English noun (NOUN), a mathematical expression (EXP), or a sentence (SEN).

♣ In any sentence, circle the verb.

♣ Classify the truth value of any entry that is a *sentence*: TRUE (T), FALSE (F), or SOMETIMES TRUE/SOMETIMES FALSE (ST/SF). The first one is done for you.

1.	$a + b = b + a$	SEN, T
2.	$a + b$	________
3.	$a + b = 5$	________
4.	rectangle	________
5.	Every rectangle has three sides.	________
6.	$x + (-x) \neq 0$	________
7.	$3 \leq 3$	________
8.	$y \geq y$	________
9.	$y > y + 1$	________
10.	$y > y - 1$	________
11.	Bob	________
12.	Bob has red hair.	________
13.	For all nonzero real numbers x, $x^0 = 1$.	________
14.	The distance between real numbers a and b is $b - a$.	________
15.	$a(b + c)$	________
16.	$a(b + c) = ab + ac$	________

★★

Do you really want to be reading this? Remember that the symbol ★★ means that the material is probably appropriate for the instructor's eyes only.

Experienced mathematicians tend to regard sentences with variables as implicit generalizations; thus a mathematician who reads

'The distance between real numbers a and b is $b - a$.'

will automatically interpret it as

'For all real numbers a and b, the distance between a and b is $b - a$.'

The latter sentence is, of course, false. However, at this point in the text, the student is expected to view the sentence

'The distance between real numbers a and b is $b - a$.'

as being sometimes true, sometimes false (depending on the choices made for a and b).

EXERCISE 4

♣ 1. Use the English noun 'Julia' in three sentences: one that is true, one that is false, and one whose truth cannot be determined without additional information.

♣ 2. Use the expression $x^2 + y^2$ in three mathematical sentences: one that is (always) true, one that is (always) false, and one that is sometimes true/sometimes false.

read mathematics out loud

After you write any mathematics (perhaps you have solved a homework problem) you should read it back to yourself, out loud, and be sure that

- it expresses a complete thought;
- it expresses a correct thought.

ALGEBRA REVIEW

Greek letters, the real numbers

Greek letters

Mathematicians are extremely fond of Greek letters. Here are some that are most commonly used, together with their names.

uppercase	lowercase	Name of Greek letter	Pronunciation
	α	alpha	AL-fa
	β	beta	BĀ-ta
Γ	γ	gamma	GAM-a
Δ	δ	delta	DEL-ta
	ϵ	epsilon	EP-si-lon
	θ	theta	THĀ-ta
	λ	lambda	LAM-da
	μ	mu	mew
	π	pi	pie
	ρ	rho	row
	τ	tau	Ow! with a 't' in front
	ϕ	phi	fee
	ω	omega	o-MĀ-ga

EXERCISE 5

learn the Greek letters

♣ Learn to recognize and name all the Greek letters listed in the table above. Practice writing them. Learn how to correctly pronounce each name (ask your instructor if you're uncertain). Once you think you have them mastered, test yourself by filling in the blanks below.

uppercase	lowercase	Name of Greek letter
	______	alpha
	______	beta
______	______	gamma
______	______	delta
	______	epsilon
	______	theta
	______	lambda
	______	mu
	______	pi
	______	rho
	______	tau
	______	phi
	______	omega

EXERCISE 6

$\mathbb{R}$

the real numbers

The *real numbers*, denoted by the symbol $\mathbb{R}$, can most easily be understood in terms of a number line:

$$\longleftrightarrow$$

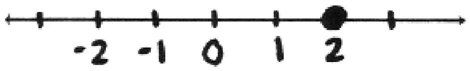
(The arrows suggest that the line extends infinitely far in both directions.) This line is a conceptually perfect picture of the real numbers in the following sense:

- Every point on this line is uniquely identified with a real number.
- Every real number is uniquely identified with a point on this line.

It is important to realize that a particular real number *may have lots of names*: for example,

$$2,\ 3^2 - 7,\ \frac{4}{2},\ 5 - 3,\ \frac{2\pi}{\pi},\ \text{and}\ \frac{-11.4}{-5.7}$$

are all names for the unique number shown below:



The name that we choose to use for a number depends on what we are doing. For example, we will see that when talking about the slope of a line, the 'names' $\frac{2}{1}$ or $\frac{-4}{-2}$ are often more useful than the shorter name 2.

♣ 1. Give ten more 'names' for the number 5.

♣ 2. Give three more 'names' for the number 2.3 .

EXERCISE 7

positive

negative

'negative' versus 'minus'

The real numbers to the right of zero on the number line are called *positive*. For example, the numbers 3, $\frac{1017}{23}$, 0.00023 and 10^{52} are positive. The real numbers to the left of zero are called *negative*. The number 0 (read this as 'zero', *not* 'oh') is not positive or negative.

The symbol '$-$' is read differently depending upon the context. If the symbol '$-$' is being used to denote a negative number, it is read as *negative*:

'-3' is read as *negative three*.

If the symbol $-$ is being used to denote the operation of subtraction, it is read as *minus*:

'$3 - 5$' is read as *three minus five*.

Here's an example that uses both:

'$3 - (-5)$' is read as *three minus negative five*.

♣ 1. How would you read '$-4 - (-3)$'?

♣ 2. What is the least positive number that can be represented on your hand-held calculator? Call it L. What do you get when you use your calculator to compute $L/2$?

♣ 3. What is the greatest positive number that can be represented on your hand-held calculator? Call it G. What happens when you use your calculator to compute $(G + 1) - G$?

EXERCISE 8
inequality symbols
$>$, $<$

One extremely nice property of the real numbers is that they are *ordered*. This means the following: given any two real numbers, either

- they are *equal*, or
- one of the numbers lies further to the left on the real number line.

When a number n lies to the left of a number m, we write

$$n < m$$

and read this as

$$n \text{ is less than } m.$$

Never read this as 'n is smaller than m'. Why not? Well, consider the numbers -5 and 3. Certainly -5 lies to the left of 3, so that $-5 < 3$ (read as *negative five is less than three*). But would you really want to say that -5 is 'smaller than' 3?

♣ 1. Comment. That is, why might you be uncomfortable saying that -5 is 'smaller' than 3?

Similarly, when n lies to the right of m, we write

$$n > m$$

and read this as

$$n \text{ is greater than } m.$$

Again, *never read this as* 'n is bigger than m'.

♣ 2. Find numbers n and m for which $n > m$ is true, but you would feel uncomfortable saying that n is 'bigger' than m.

♣ 3. Read the following sentences out loud, and determine if they are TRUE or FALSE:

a) $-3 < 2$
b) $\frac{2}{5} > \frac{3}{7}$
c) $-3 > -7$

EXERCISE 9
reread the section

♣ Reread this section. You will need to read each section in this text *at least twice* to fully understand the material. Also, don't expect to read mathematics the same way that you read English. You're probably used to measuring reading rates in units of 'pages per hour'. Mathematics is read in units of 'hours per page'.

QUICK QUIZ
sample questions

1 What is the mathematical analogue of an English noun?
2 In English, a noun is usually a person, place, or thing. List three common types of mathematical 'nouns'.
3 Use the mathematical expression x in a sentence that is always true.
4 Circle the entries that are *sentences*:

$$\frac{2}{y} - 1 \qquad \sqrt{x} > 2 \qquad 4 - 3 = 7$$

KEYWORDS
for this section

English noun, mathematical expression, sentences, Greek letters, the real numbers, $\mathbb{R}$*, positive, negative, minus, inequality symbols.*

END-OF-SECTION EXERCISES

Classify each entry in the list below as: an expression (EXP), or a sentence (SEN).

In any sentence, circle the verb.

Classify the truth value of any entry that is a *sentence*: TRUE (T), FALSE (F), or SOMETIMES TRUE/SOMETIMES FALSE (ST/SF).

NOTE: The symbol '$\approx$' means '*is approximately equal to*'.

1. $\frac{1}{3}$
2. π
3. $\frac{1}{3} = 0.\overline{3}$
4. $\frac{1}{3} = 0.3\overline{3}$
5. $\frac{1}{3} = 0.33$
6. $\frac{1}{3} = 0.33333$
7. $\frac{1}{3} \approx 0.33$
8. $\frac{1}{3} \approx 0.333333$
9. $x^2 > 0$
10. $y^2 > 0$
11. $x^2 \geq 0$
12. $y^2 \geq 0$
13. $(-3)(-5)$
14. $(-3) + (-5)$
15. $-5 < -3$
16. $-3 < -5$
17. $|t| > 0$ (Need help with absolute values? You might want to skip ahead to the Algebra Review in Section 2.1)
18. $|x| > 0$
19. $|t| \geq 0$
20. $|x| \geq 0$
21. $|t| < 0$
22. $|x| < 0$
23. $|3 - \pi| = \pi - 3$
24. $|\pi - 3| = \pi - 3$
25. $|t| = t$
26. $|t| = -t$
27. $\frac{x}{y} \div \frac{z}{w}$
28. $\frac{x}{y} \cdot \frac{w}{z}$
29. $\frac{x}{y} \div \frac{z}{w} = \frac{xw}{yz}$
30. $\frac{x}{y} \cdot \frac{w}{z} = \frac{xw}{yz}$
31. $a(bc) = (ab)c$
32. $a + (b + c) = (a + b) + c$
33. $3x^2 = (3x)^2$
34. $(2 \cdot 3)^2 = 2 \cdot 3^2$
35. $\sqrt{(-3)^2} = -3$
36. $\sqrt{(-3)^2} = 3$

END-OF-SECTION EXERCISES

(continued)

37 The sentence below is TRUE. What name does this result usually go by? (Dust off your algebra book.)

For all real numbers a and b, $a + b = b + a$.

38 The sentence below is TRUE. What name does this result usually go by?

For all real numbers a and b, $ab = ba$.

39 The sentence below is TRUE. What name does this result usually go by?

For all real numbers a, b, and c, $a(b + c) = ab + ac$.

40 The sentence below is TRUE. What name does this result usually go by?

For all real numbers a, b, and c, $a(bc) = (ab)c$.

41 To mathematicians, subtraction is just a special kind of addition, since for all real numbers x and y,

$$x - y = x + (-y) \ .$$

That is, to subtract a number is the same as to add the opposite of the number.

♣ What is this result telling you if $x = 1$ and $y = 3$? How about if $x = 1$ and $y = -3$?

42 (Refer to the previous exercise.) To mathematicians, division is just a special kind of multiplication. WHY? Be sure to answer in a complete sentence.

43 (Importance of ASSOCIATIVE laws) The word 'associative' has the same root as the English words *sociable* and *associate*. These English words have to do with groups (e.g., a *sociable* person is one who enjoys being in a group of people). Thus it should not be surprising that 'associative' laws in mathematics have to do with grouping.

The *associative law of multiplication* states: for all real numbers x, y and z, $x(yz) = (xy)z$. Thus, the grouping of numbers in a product is irrelevant. It is *because of this property* that we are able to write xyz, with no parentheses.

♣ Using complete sentences, comment on why the associative law makes expressions like xyz unambiguous.

44 (Refer to the previous exercise.) What property of the real numbers allows us to write things like $a + b + c$ with no ambiguity?