2.2 Graphs of Functions

Introduction

Associated with every function is a *set* called *the domain of the function.* This set influences what the graph of the function looks like.

DEFINITION *domain of* f, $\mathcal{D}(f)$	The set of inputs to a function f is called the *domain of* f, and denoted by $\mathcal{D}(f)$.

the domain convention

The *domain convention* says the following: if the domain of a function is not explicitly specified, then it is assumed to be all inputs for which the function makes sense. Things to watch for:

- division by zero is not allowed
- numbers under even roots ($\sqrt{\ }$, $\sqrt[4]{\ }$, $\sqrt[6]{\ }$, etc.) must be nonnegative
- 0^0 is not defined

There is a very convenient notation for functions, to be discussed later on in this section, that explicitly shows the domain. This is useful when we want to take the domain to be different than the set dictated by the domain convention.

EXAMPLE

using the domain convention, function of one variable

Consider the function given by $f(x) = \frac{\sqrt{x+1}}{x+2}$. No domain is specified for the function, so the *domain convention* is used to determine the domain. The expression under the radical must be nonnegative, and $x+2$ cannot equal zero. Thus:

$$\begin{aligned}\mathcal{D}(f) &= \{x \mid x+1 \ge 0 \text{ and } x+2 \ne 0\}\\ &= \{x \mid x \ge -1 \text{ and } x \ne -2\}\\ &= \{x \mid x \ge -1\} \qquad (*)\end{aligned}$$

Note that if $x \ge -1$ is true, then automatically $x \ne -2$ is true. So in this case:

$$(\, x \ge -1 \text{ and } x \ne -2\,) \iff x \ge -1$$

By the *domain convention*, the domain of $f(x) = \frac{\sqrt{x+1}}{x+2}$ is (using interval notation) $[-1, \infty)$.

another use of the '=' sign; equality of sets

In the example above, the answer was written down using a *complete mathematical sentence*. The '=' signs used in (*) are for *equality of sets*: two sets are equal when they have the same members. For example, $A = \{1,2,3\}$ and $B = \{2,3,1\}$ are equal sets. The order in which the elements are listed is unimportant.

So far, you have seen the equal sign ('=') used in two different contexts: equality of *numbers* (or, expressions representing numbers), and equality of *sets*. You must be able to recognize, from context, the proper interpretation of an '=' sign.

EXAMPLE

using the domain convention, function of two variables

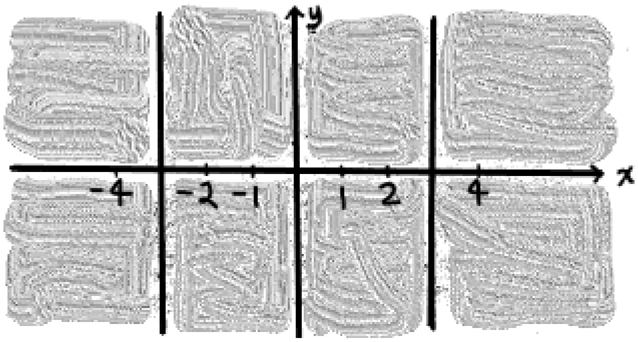
Consider the function given by $g(x,y) = \frac{1}{x^2-9} + \frac{1}{xy}$. If no domain is specified, we need only exclude inputs (x,y) for which the function does not make sense. The function is not defined if $x^2 - 9 = 0$; also, it is not defined if $xy = 0$. So the following points must be *excluded*:

$$\begin{aligned}\{(x,y) \mid x^2 - 9 = 0\} &= \{(x,y) \mid x^2 = 9\} \\ &= \{(x,y) \mid x = 3 \text{ or } x = -3\} \\ &= \{(3,y) \mid y \in \mathbb{R}\} \cup \{(-3,y) \mid y \in \mathbb{R}\}\end{aligned}$$

and

$$\begin{aligned}\{(x,y) \mid xy = 0\} &= \{(x,y) \mid x = 0 \text{ or } y = 0\} \\ &= \{(0,y) \mid y \in \mathbb{R}\} \cup \{(x,0) \mid x \in \mathbb{R}\}\end{aligned}$$

The domain of the function g is the portion of the xy-plane that remains after the necessary points are excluded. This domain is shaded below.



EXERCISE 1

interpreting a mathematical sentence

Analyze the mathematical sentence that appears in the example above:

$$\begin{aligned}\{(x,y) \mid x^2 - 9 = 0\} &= \{(x,y) \mid x^2 = 9\} && \text{(line 1)} \\ &= \{(x,y) \mid x = 3 \text{ or } x = -3\} && \text{(line 2)} \\ &= \{(3,y) \mid y \in \mathbb{R}\} \cup \{(-3,y) \mid y \in \mathbb{R}\} && \text{(line 3)}\end{aligned}$$

The lines have been numbered for easy reference.

♣ 1. Seven equals signs appear in this mathematical sentence. Which are being used for equality of numbers? Which are being used for equality of sets?

♣ 2. What allows the replacement of $x^2 = 9$ by ($x = 3$ or $x = -3$) in going from line 1 to line 2?

♣ 3. What does the symbol $\cup$ mean in line 3?

♣ 4. Why is line 2 equal to line 3?

EXERCISE 2
finding domains

Use the *domain convention* to find the domains of the following functions. Be sure to write *complete mathematical sentences.* Show the domains on a number line, or in the xy plane, whichever is appropriate.

♣ 1. $f(x) = \frac{\sqrt{x-1}}{x+2}$

♣ 2. $g(x) = \frac{\sqrt[3]{x-1}}{x^2-4}$

♣ 3. $h(x,y) = \frac{\sqrt{x}}{x+y}$

Now, we are in a position to define the *graph of a function.* First, the *graph of a function of one variable*:

DEFINITION
the graph of a function of one variable

Let f be a function of one variable. Then:

$$\text{the graph of } f = \{(x, f(x)) \mid x \in \mathcal{D}(f)\}$$

That is, the graph of f consists of points of the form (input, output), where x is the input, and $f(x)$ is the corresponding output.

In other words, the graph of f is the same as the graph of the equation $y = f(x)$. Merely plot the function values $f(x)$ as the y-values, and proceed as earlier.

slight abuse of notation

Actually, most people think of the *graph of* f as a (partial) *picture* of the set $\{(x, f(x) \mid x \in \mathcal{D}(f)\}$. Since there is such a close association between the set of points, and the *picture* of the set of points, this should cause no confusion.

EXAMPLE
graphing a function of one variable

Problem: Graph the function defined by $f(x) = \frac{1}{x-1}$.

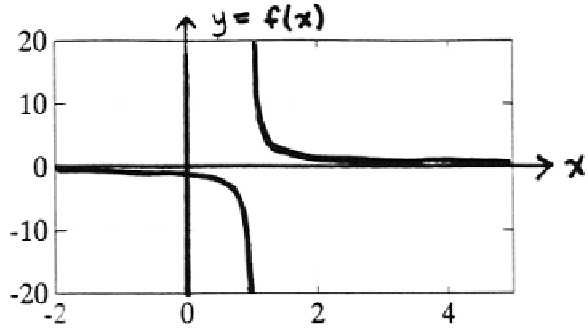
Solution: By the domain convention:

$$\mathcal{D}(f) = \{x \mid x - 1 \neq 0\} = \{x \mid x \neq 1\}$$

Then:

$$\begin{aligned}\text{the graph of } f &= \{(x, f(x)) \mid x \in \mathcal{D}(f)\}\\ &= \{\big(x, \frac{1}{x-1}\big) \mid x \neq 1\}\end{aligned}$$

The graph is shown below. It is of course impossible to show *all* points of the form $(x, f(x))$, since x is allowed to take on values in $(-\infty, 1) \cup (1, \infty)$. It is customary to show the interesting part(s) of the graph. A mathematician looking at this graph would assume that the pattern displayed near the graph boundaries would continue ad infinitum.



EXAMPLE
a 'punctured' graph

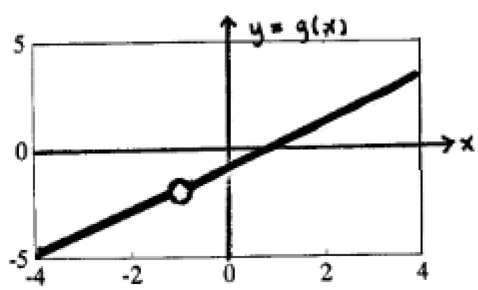
Problem: Graph $g(t) = \frac{t^2-1}{t+1}$.

Solution: Note that t cannot equal -1, for this would produce division by zero. But in the same breath you must notice that the numerator is *also* zero when $t = -1$. Since -1 is a zero of $t^2 - 1$, this means that $(t - (-1))$ is a factor of $t^2 - 1$ (see the Algebra Review, this section). Indeed, factoring yields $t^2 - 1 = (t + 1)(t - 1)$. For values of t different from -1, the function has a simpler expression:

$$\frac{t^2 - 1}{t + 1} = \frac{(t + 1)(t - 1)}{t + 1} \overset{t \neq -1}{=} t - 1$$

Note that the expressions $\frac{t^2-1}{t+1}$ and $t - 1$ are NOT exactly the same! They *are* the same a great deal of the time; whenever t is not -1. But when t is -1, they act differently: $\frac{t^2-1}{t+1}$ is not defined, but $t - 1$ is perfectly well defined, and equals $(-1) - 1 = -2$.

The graph of g is shown below.



EXAMPLE
more on 'puncturing' a graph

Problem: Write a formula for a function whose graph is the same as the graph of $f(x) = x^2 + 2$, but is 'punctured' where $x = 3$.

Solution: $P(x) = (x^2 + 2) \cdot \frac{x-3}{x-3} = \frac{x^3-3x^2+2x-6}{x-3}$

EXERCISE 3
graphing functions of one variable

Graph the following functions:

♣ 1. $f(x) = \sqrt{x} - 2$

♣ 2. $g(t) = 2|t| - 1$

♣ 3. $h(\omega) = \frac{\omega^2+\omega-6}{\omega+3}$

DEFINITION
the graph of a function of two variables

Let f be a function of two variables. Then:

$$\text{the graph of } f = \{(x, y, f(x, y)) \mid (x, y) \in \mathcal{D}(f)\}$$

Since the graph of a function of two variables is a set of points *in space*, it is more difficult to draw. We will restrict ourselves to graphing only functions of one variable.

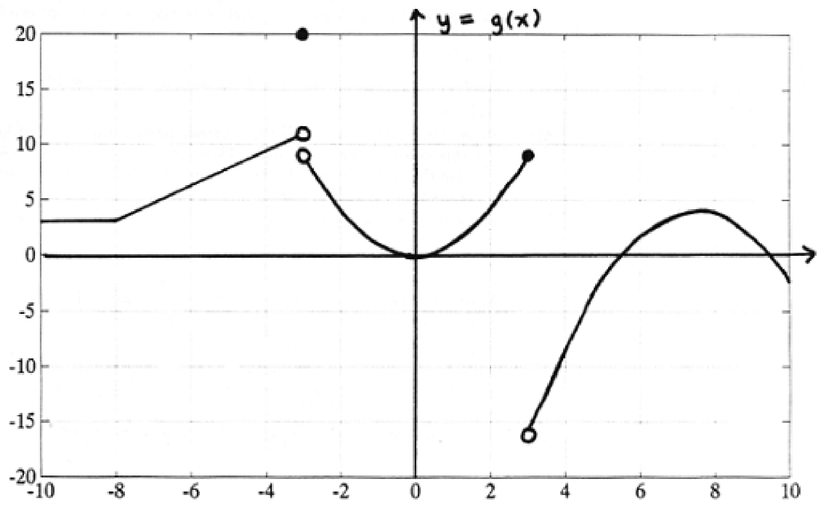
EXERCISE 4

♣ What do you suppose is the definition of *the graph of a function of three variables*?

reading information off a graph

Often, you will be *given* the graph of a function and asked to read information off the graph.

For example, consider the graph of a function g, shown below.



questions about the graph

Notice first that the graph is labeled $y = g(x)$. This tells you that the y values on the graph are the function values from a function g.

You could be asked the following questions:

1) How can you confirm that this is indeed the graph of a *function*?
2) What is $g(-10)$?
3) What is $g(-9.2)$?
4) What is $g(-3)$?
5) What is $g(0)$?
6) Find: $\{x \mid g(x) = 10\}$
7) Find: $\{x \mid g(x) = 3\}$
8) Find: $\{x \mid g(x) \ge 0\}$
9) Based on this graph, what would you suspect that $g(-11)$ is?
10) Based on this graph, what would you suspect that $g(20)$ is?

the answers

When answering these questions, be sure to write *complete mathematical sentences.*

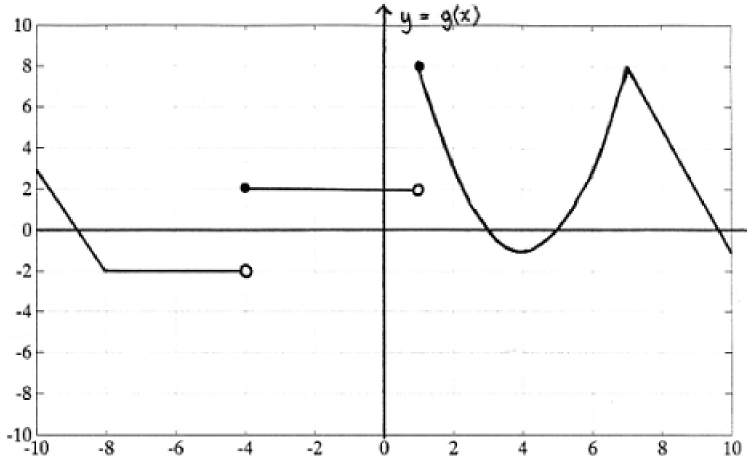
1) The graph passes the vertical line test. Every input has associated to it a *unique* output.
2) $g(-10) = 3$. **Don't just give the answer as: 3!** When reading information off a graph, it may be necessary to use your judgment and estimate. Perhaps it would be better to say $g(-10) \approx 3$; here, the symbol '$\approx$' means 'is approximately equal to'. Most people just use the '$=$' sign, with the understanding that there may be some error involved in reading off the graph.
3) $g(-9.2) = 3$
4) $g(-3) = 20$. Note that *the dot is filled in at the y value of* 20.
5) $g(0) = 0$
6) There is only one x value where the corresponding y value is 10; it looks like this occurs when $x \approx -3.5$. Thus, $\{x \mid g(x) = 10\} = \{-3.5\}$. Observe that the number -3.5 must go in a *set*, because the '$=$' sign is being used for equality of sets.
7) There are lots of x-values with corresponding y-values equal to 3 :

$$\{x \mid g(x) = 3\} = [-10, -8] \cup \{-1.7, 1.7, 6.5, 8.5\}$$

Again, it is necessary to approximate. Recall that $[-10, -8]$ is a *set*. The union symbol $\cup$ is used to 'join together' the two sets.

8) $\{x \mid g(x) \ge 0\} = [-10, 3] \cup [5.5, 9.5]$
9) Assuming that the 'interesting' part of the graph is shown, and that the patterns indicated on the boundaries of the graph would continue, one would estimate that $g(-11) = 3$.
10) This author would estimate that $g(20)$ is a large negative number.

EXERCISE 5 Answer the following questions about the graph of the function g shown below. It may be necessary to estimate.



♣ 1. How can you confirm that this is indeed the graph of a *function*?

♣ 2. What is $g(-10)$?

♣ 3. What is $g(-6.4)$?

♣ 4. What is $g(-4)$?

♣ 5. What is $g(1)$?

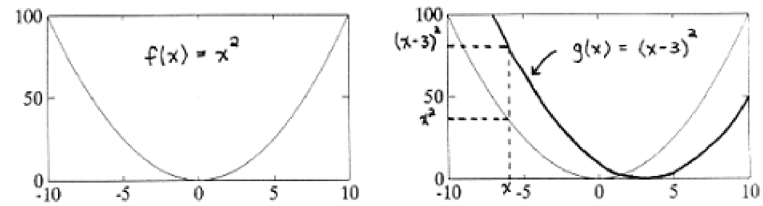
♣ 6. Find: $\{x \mid g(x) = 8\}$

♣ 7. Find: $\{x \mid g(x) = -4\}$

♣ 8. Find: $\{x \mid g(x) \le 0\}$

shifting graphs left and right

Consider the function f given by $f(x) = x^2$. We can define a new function g that uses the function f, by $g(x) := f(x-3) = (x-3)^2$. The graphs of f and g are shown below. Remember that the symbol ':=' is used when it is desired to emphasize that something is being *defined*.



Note that the graph of g is the same as the graph of f, except shifted three units *to the right*. This is often confusing to students: they feel that since we evaluated f at x *minus* 3, the graph ought to shift to the *left*. Let's investigate what's really happening here.

FACT:
shifting a function to the right

FACT: Let f be a function, and let $c > 0$. Define a new function g by:

$$g(x) := f(x-c)$$

Then, the graph of g is the same as the graph of f, except shifted c units to the right.

REASONING:

REASONING: The first question that needs to be answered is: "What is the domain of g?" For $g(x)$ to make sense, f must know how to act on $x-c$. Thus, $\mathcal{D}(g) = \{x \mid x - c \in \mathcal{D}(f)\}$. Then:

$$\begin{aligned}
\text{the graph of } g &= \{(x, g(x)) \mid x \in \mathcal{D}(g)\} && \text{(defn. of the graph of } g\text{)}\\
&= \{(x, f(x-c)) \mid x - c \in \mathcal{D}(f)\} && \text{(defn. of } g(x)\text{)}\\
&= \{(z+c, f(z)) \mid z \in \mathcal{D}(f)\} && \text{(define } z = x-c\text{, so } x = z+c\text{)}\\
&= \{(x+c, f(x)) \mid x \in \mathcal{D}(f)\} && \text{(change dummy variable)}
\end{aligned}$$

you must understand every line of this sentence

It is important that you understand every line of this mathematical sentence.

In particular, what happened in going from line 2 to line 3? It was desired to have a *single* variable as the input to f, instead of $x-c$. This was accomplished by *defining* z to be $x-c$. (Instead of the name 'z', we could have used ω, or t, or s, or $\ldots$.) Then, everything was rewritten in terms of z.

What happened in going from line 3 to line 4? Not really anything! Students are often more comfortable working with the variable x than the variable z; this alone was the motivation for changing the name of the dummy variable.

Be sure to understand that the x in line 1 *has nothing to do with* the x in, say, line 4. In line 1, x represents a typical element of the domain of g. In line 4, x represents a typical element of the domain of f.

It is important that you understand every line of this mathematical sentence. The next two exercises check your understanding.

EXERCISE 6

♣ Let f be a function, and let $c > 0$. Define a new function g by $g(x) := f(x+c)$. Prove that the graph of g is the same as the graph of f, except shifted c units *to the left*. Be sure to write complete mathematical sentence(s).

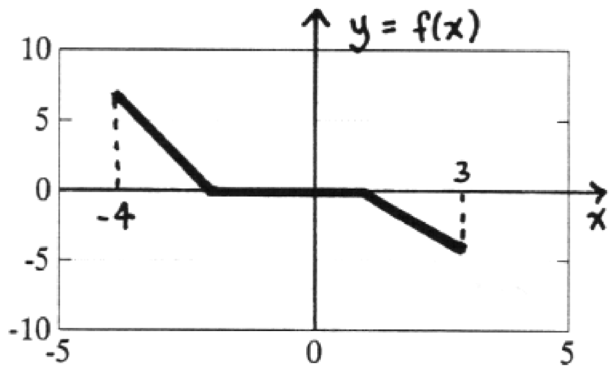
EXERCISE 7

The graph of a function f is shown below.

♣ 1. What is the domain of f?

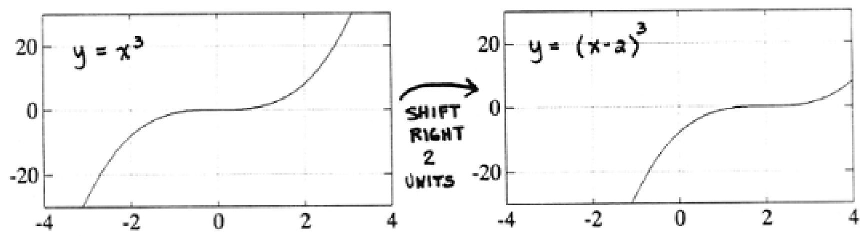
♣ 2. Define a new function h by $h(x) := f(x-2)$. What is the domain of h? Graph h.

♣ 3. Define a new function g by $g(x) := f(x+3)$. What is the domain of g? Graph g.

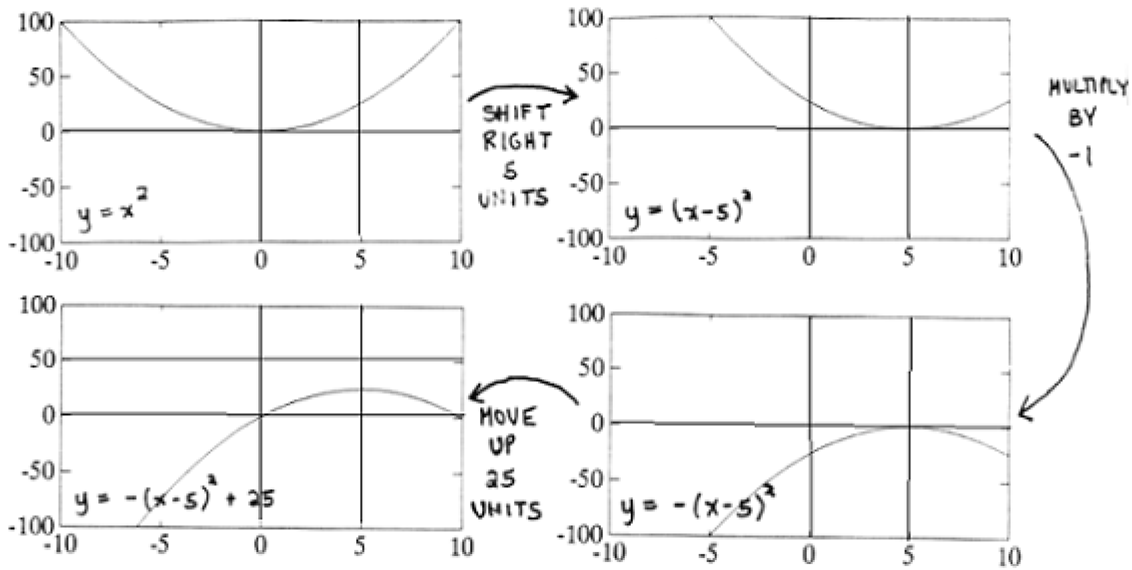


more on building graphs from simpler pieces

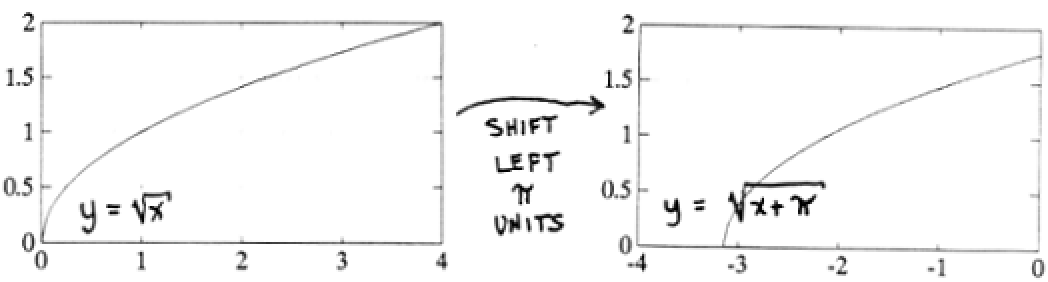
The following sketches illustrate how graphs can often be 'built up' from simpler pieces. This technique was investigated in an earlier section; some slightly more complicated examples appear here.



Graphing $f(x) = (x-2)^3$



Graphing $f(x) = -(x-5)^2 + 25$



Graphing $f(x) = \sqrt{x + \pi}$

convenient function notation that explicitly shows the domain

$f\colon A \to B$

Occasionally, one wants a function to have a domain that is *different* from its ‘natural domain’, that is, the domain dictated by the domain convention. In particular, if you are using a computer to graph a function, you must often restrict yourself to a set much smaller than the natural domain.

The function notation

$$f\colon A \to B$$

is particularly convenient in such cases. Read $f\colon A \to B$ as ‘f, from A to B’. Here is the meaning of each part of this symbol.

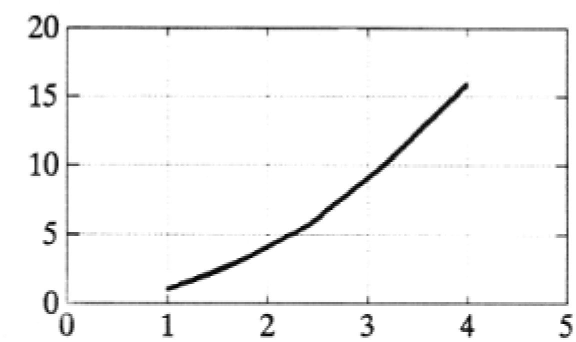
explaining the notation $f\colon A \to B$

- The first letter that appears (here, f) is the *name of the function.*
- The colon ': ' separates the function name from the rest of the symbol.
- The first letter after the colon (here, A) is the *domain of the function.* Thus, A is a *set* that contains the inputs to f.
- The arrow '$\to$' suggests that the inputs from A are being 'sent to' outputs in B.
- The last letter (here, B) can be thought of as the *output set.* It is used to answer the question: "What sort of outputs do we get when f acts on the elements from A?" In this course, the outputs of our functions will always be real numbers, so we can always let B be the real numbers, $\mathbb{R}$.
- Note that the notation $f\colon A \to B$, by itself, does *not* tell the rule that f uses to go from the inputs in A to the outputs in B. Thus, this notation *must be accompanied by a rule*, as illustrated in the examples below.

EXAMPLE

Graph: $f\colon [1,4] \to \mathbb{R},\ f(x) = x^2$

The 'natural domain' of the function f given by the rule $f(x) = x^2$ would be all real numbers, since all real numbers can be squared. Here, we want to instead take the domain to be $[1,4]$. Thus, the graph is:

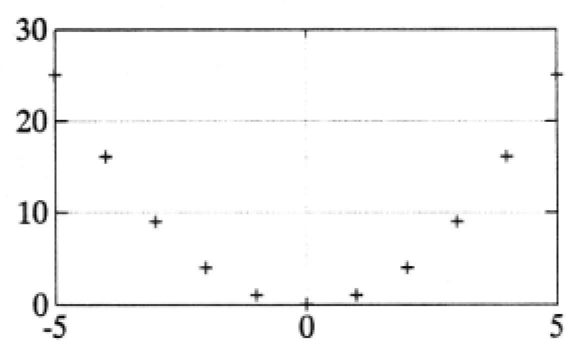


EXAMPLE

Graph the function g defined by:

$$g\colon \mathbb{Z} \to \mathbb{R},\ g(x) = x^2$$

The (partial) graph of g is shown below.



EXERCISE 8

Graph the following functions:

♣ 1. $f\colon [-2,1] \to \mathbb{R},\ \ f(x) = x^3$

♣ 2. $g\colon [-2,-1] \cup [1,2] \to \mathbb{R},\ \ g(x) = x^2$

♣ 3. $h\colon \{-3,-2,-1,0,1,2,3\} \to \mathbb{R},\ \ h(x) = x+1$

ALGEBRA REVIEW

factoring

To *factor an expression* means to take a *sum* (things added) and write it as a *product* (things multiplied).

For example, $x^2 + 2x - 3 = (x-1)(x+3)$. These two expressions are equal for all real numbers x. The expression $(x-1)(x+3)$ is said to be in *factored form.* The process of going from the sum $x^2 + 2x - 3$ to the product $(x-1)(x+3)$ is the process of *factoring.* You studied lots of techniques for factoring in algebra.

polynomial

Recall that a *polynomial* (in one variable x) is a sum of terms, where each term is of the form ax^n. Here, a is any real number, and $n \in \{0,1,2,3,\dots\}$. For example, $2x^3 - x^2 - 2x + 1$ is a polynomial (note that 1 can be written as $1 \cdot x^0$).

zero (root) of a polynomial

A *zero* (or *root*) of a polynomial is a *number that makes the polynomial equal to zero.* For example, 1 is a zero of $P(x) := 2x^3 - x^2 - 2x + 1$, because:

$$P(1) = 2(1)^3 - (1)^2 - 2(1) + 1 = 2 - 1 - 2 + 1 = 0$$

Also, $1/2$ is a root because $P(1/2) = 0$. (♣ Check this.)

relationship between the zeros and factors of a polynomial

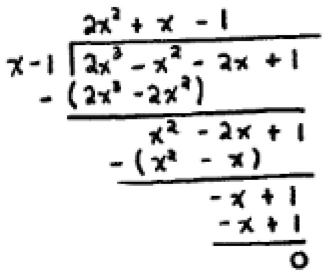
There is a fundamental relationship between the zeros of a polynomial, and the factors of the polynomial. Let $P(x)$ denote any polynomial in x.

- If r is a zero of P (so that $P(r) = 0$), then $x - r$ is a factor of P.
- And, if $x - r$ is a factor of P, then r is a zero of P.

We can sometimes use this fact, together with long division, to help us factor polynomials.

EXAMPLE

For example, we saw that 1 is a root of $2x^3 - x^2 - 2x + 1$. Thus, $x - 1$ is a factor. That is, $x - 1$ must 'go into' $2x^3 - x^2 - 2x + 1$ evenly. Do a long division:



Now we know that $2x^3 - x^2 - 2x + 1 = (x-1)(2x^2 + x - 1)$. (♣ Finish factoring this polynomial.)

EXERCISE 9

using a zero to factor a polynomial

Check that the given number is a zero of the polynomial. Then, use this zero to get a factor. Do a long division to get another factor. Factor each polynomial as completely as possible.

♣ 1. $P(x) = x^3 + x^2 - 9x - 9$; -1

♣ 2. $P(x) = 2x^3 - 3x^2 - 11x + 6$; 3

QUICK QUIZ

sample questions

1. Use the domain convention to find the domain of $f(x) = \frac{\sqrt{2x-1}}{x^2-9}$. Be sure to write a complete mathematical sentence.
2. True or False: $\{1, 2, 3\} = \{3, 2, 1\}$. In this sentence, the '=' sign is being used for equality of ___________ .
3. What is the graph of a function f of one variable? Be sure to answer in a complete mathematical sentence.
4. Graph $f(t) = 2\sqrt{t-3}$ by building it up from 'simpler pieces'. What is $\mathcal{D}(f)$?
5. Verify that -1 is a root of $P(x) = x^4 - 2x^2 + 1$. From this information, get a factor of P. Then, use long division to get a second factor.

KEYWORDS

for this section

Domain of a function f, domain convention, two uses for the '=' sign, the graph of a function of one variable, the graph of a function of two variables, zero (root) of a polynomial, factoring polynomials, shifting graphs left and right, building graphs of functions up from simpler pieces, the $f\colon A \to B$ function notation.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE (T), FALSE (F), or CONDITIONAL (C).

1. $\{x \mid x + 1 \ge 0\}$
2. $\mathcal{D}(f) = \{x \mid x + 1 \ge 0\}$
3. $(x \ge 2 \text{ and } x \ne 1) \iff x \ge 2$
4. $(x \ge 2 \text{ and } x \ne 3) \iff x \ge 2$
5. If P is a polynomial in x, and $P(-3) = 0$, then $x + 3$ is a factor of P.
6. $\{x \mid x = 3\} = \{y \mid y = 3\}$
7. $\{x \mid x = 3\} = \{y \mid y - 3 = 0\}$
8. Define f by $f(x) = \frac{\sqrt{x-1}}{x+2}$.
9. The graph of g is $\{(t, g(t)) \mid t \in \mathcal{D}(g)\}$.
10. $f\colon [-3, 1] \to \mathbb{R}$

Graph the following functions.

11. $f\colon [-1, 1] \to \mathbb{R}$, $f(x) = (x + 1)^3$
12. $g\colon (-\infty, 4] \to \mathbb{R}$, $g(t) = 3|t - 2| - 1$
13. $h\colon \{1, 4, 9, 16, 25\} \to \mathbb{R}$, $h(t) = \sqrt{t}$
14. $f : \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$