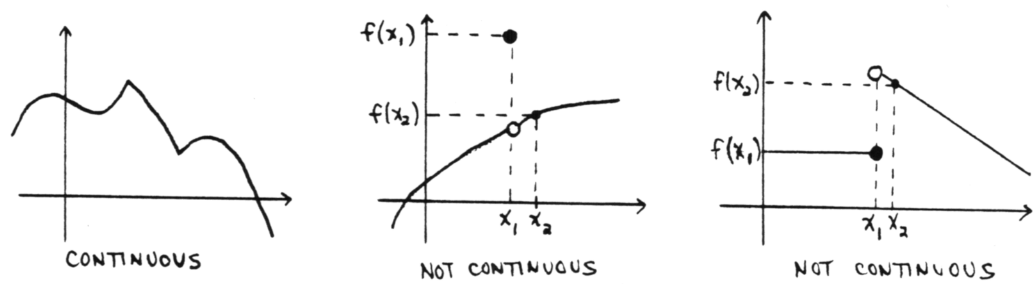
3.1 Limits—The Idea

continuity;
when inputs are 'close', corresponding outputs are 'close'

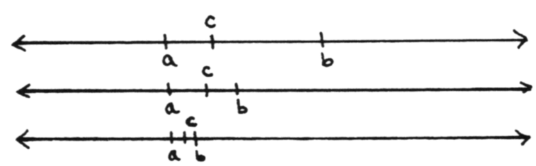
Certain functions have the property that when their inputs are close, so are their outputs. The mathematical idea that addresses this issue is *continuity*. Intuitively, a function is *continuous* if its graph can be traced without picking up your pencil; it can't have any 'breaks'. In other words, for a continuous function, when inputs are 'close together' the corresponding outputs should be 'close together'. This certainly doesn't happen in the second and third sketches below: in both cases, x_1 is 'close to' x_2, but $f(x_1)$ is not 'close to' $f(x_2)$.



What is meant by numbers being 'close'?

The idea of 'closeness' is not precise, at least in the English sense. Are the numbers 2 and 3 'close'? How about 2 and 2.01? How about 2 and 2.00001?

Just how 'close' can two different numbers be? The answer is really quite simple: *as close as you want*. The real numbers have a beautiful property: given any two real numbers a and b, if they're not equal, then there's another real number between them.



the mathematical tool that addresses the idea of 'numbers being close' is the ***limit***

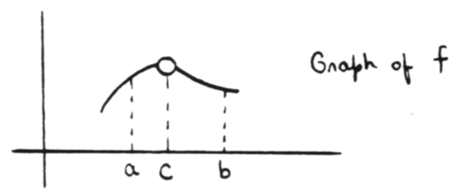
In order to discuss *continuity*, it is first necessary to have a mathematical tool that addresses, precisely, the notion of 'numbers being close'. The tool that accomplishes this is the mathematical *limit*. In our first—informal—discussion of limits, the word 'close' will be used in an intuitive sense. However, in the next section, you will see how this notion of 'closeness' is addressed precisely.

some initial assumptions about the functions we're working with

Suppose, for now, that any function f we're working with has the following property: it is defined near a number c, but not necessarily at c. We want to guarantee that there are inputs arbitrarily close to c (on both sides) that f knows how to act on. This requirement can be phrased more precisely—there must be numbers a and b such that:

$$(a,c) \cup (c,b) \subset \mathcal{D}(f)$$

This requirement will be weakened when things are made precise in the next section.



the mathematical sentence, $\lim\limits_{x\to c} f(x) = l$

The mathematical sentence

$$\lim_{x\to c} f(x) = l \qquad (*)$$

is read as:

The limit of $f(x)$, as x approaches c, is equal to l.

This sentence involves a function f, a constant c, and a constant l. The 'x' that appears twice (once in '$x \to c$', and once in '$f(x)$') is a dummy variable; it could equally well be called 't' or 'y' or 'ω'. As you'll see momentarily, x represents a number that is getting closer and closer to c.

The sentence can be true or false. We will be primarily interested in cases when it is true.

When is the sentence $\lim\limits_{x\to c} f(x) = l$ *true?*

In order for the mathematical sentence (*) to be true, the following two conditions must be satisfied:

- as x gets close to the number c coming in from the right-hand side, the corresponding function values $f(x)$ must get close to l; and
- as x gets close to c from the left-hand side, the corresponding function values $f(x)$ must also get close to l.

Thus, in order for the sentence $\lim_{x\to c} f(x) = l$ to be true, the following condition must be satisfied: when x is close to c, $f(x)$ must be close to l.

$\lim_{x\to c} f(x) = l$, *text style*

If the sentence $\lim\limits_{x\to c} f(x) = l$ is typed in text (instead of displayed), it requires extra space between the lines, to make room for the '$x \to c$'. Lots of people think that this extra space doesn't look very good. Therefore, the sentence is usually typeset differently in text, like this: $\lim_{x\to c} f(x) = l$. The phrase $x \to c$ is moved over, merely to prevent the excess space between lines.

Evaluate the limit $\lim_{x\to c} f(x)$

You will frequently be asked to:

$$\textit{Evaluate the limit } \lim_{x\to c} f(x).$$

This means:

$$\textit{Find a number } l \textit{ so that the sentence } \lim_{x\to c} f(x) = l \textit{ is TRUE.}$$

(It will be shown that if there is such a number l, then it is unique.) If no such number l exists, then we say that:

$$\textit{The limit } \lim_{x\to c} f(x) \textit{ does not exist.}$$

EXAMPLE

finding the limit of a function

Let $f(x) = 2x$. Then:

$$\lim_{x\to 2} f(x) = 4$$

We could have instead said:

$$\textit{The mathematical sentence } \lim_{x\to 2} f(x) = 4 \textit{ is true.}$$

However, mathematicians usually have no need to say things that are false (except, perhaps, in a book on logic). Therefore, when a mathematical sentence is stated, it is assumed to be true. That is, when a mathematician states:

$$\lim_{x\to 2} f(x) = 4$$

this *means* that the sentence is true.

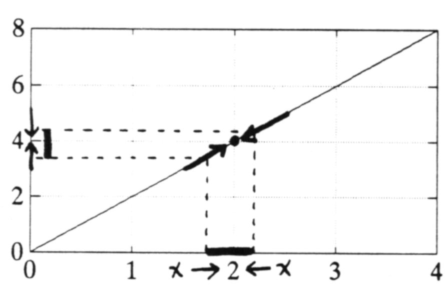
Now, why is it that this sentence is true? It is because as x approaches 2 from *either* side, the function values are getting close to 4.

For example, look at the table below. When x is 1.99, $f(x)$ is 3.98. That is, $f(1.99) = 3.98$. Also, when x is 2.001, $f(x)$ is 4.002. That is, $f(2.001) = 4.002$. These are examples of the fact that when x is close to 2, $f(x)$ is close to 4.

Now look at the graph of f, also given below. This graph clearly shows that when the inputs are close to 2, the corresponding function values are close to 4.

Note in this case that f is defined at 2, and $f(2) = 2 \cdot 2 = 4$. As x approaches 2, the corresponding function values $f(x)$ get close to $f(2)$. That is:

$$\lim_{x\to 2} f(x) = f(2)$$



GRAPH OF f,
f(x) = 2x

x	f(x)	x	f(x)
1.9500	3.9000	1.9950	3.9900
1.9600	3.9200	1.9960	3.9920
1.9700	3.9400	1.9970	3.9940
1.9800	3.9600	1.9980	3.9960
1.9900	3.9800	1.9990	3.9980
2.0100	4.0200	2.0010	4.0020
2.0200	4.0400	2.0020	4.0040
2.0300	4.0600	2.0030	4.0060
2.0400	4.0800	2.0040	4.0080
2.0500	4.1000	2.0050	4.0100

As x approaches 2,
f(x) approaches 4

EXERCISE 1

♣ 1. Let $f(x) = 2x$. Evaluate each of the following limits. Be sure to write complete mathematical sentences. (That is, if asked to evaluate $\lim_{x\to 2} f(x)$, don't just say 4. Instead, write the complete sentence: $\lim_{x\to 2} f(x) = 4$.)

$$\lim_{x\to 3} f(x) \qquad \lim_{x\to 0} f(x) \qquad \lim_{x\to \pi} f(x) \qquad \lim_{x\to 2/3} f(x) \qquad \lim_{x\to -10.1} f(x)$$

♣ 2. Let c denote a particular real number, and let $f(x) = 2x$. What is

$$\lim_{x\to c} f(x)\ ?$$

EXAMPLE

finding a limit,
f is not defined at c

Now, let $f(x) = 2x\frac{(x-2)}{(x-2)}$. In this case, f is *not defined* at $x = 2$; the graph in the previous example has been punctured.

Again:

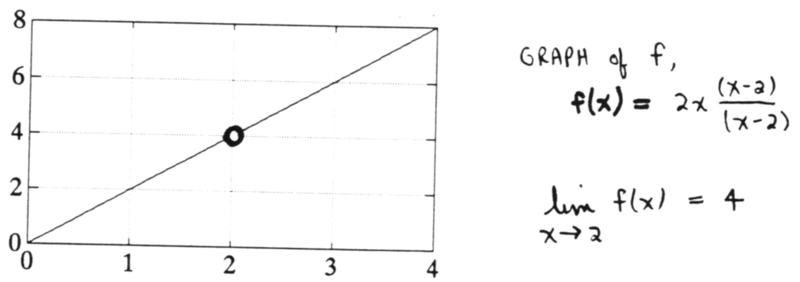
$$\lim_{x\to 2} f(x) = 4$$

This is because the two required conditions are satisfied: as x approaches 2 from the right AND the left, the corresponding function values are getting close to the number 4. That is, when x is close to 2 (but not equal to 2), the corresponding function values are close to 4.

In order to talk about the limit of a function f as x approaches c, the function f need NOT be defined at c. It need only be defined *near* c.

Here are some additional true limit statements about this function:

$$\lim_{x\to 3} f(x) = 2(3) = 6 \qquad \lim_{x\to \pi} f(x) = 2\pi \qquad \lim_{x\to 0} f(x) = 2(0) = 0$$



EXERCISE 2

♣ Evaluate the following limits. In each case, a quick sketch of the function may be helpful. Be sure to write complete mathematical sentences.

1. $\lim_{x\to 3} 2x\frac{(x-3)}{(x-3)}$ 2. $\lim_{x\to 2} 2x\frac{(x-3)}{(x-3)}$ 3. $\lim_{x\to 1} x^2\frac{(x-1)}{(x-1)}$

4. $\lim_{x\to 0} x^2\frac{(x-1)}{(x-1)}$ 5. $\lim_{x\to 3/2} \sqrt{x}\frac{(2x-3)}{(2x-3)}$

EXAMPLE

finding a limit, f is defined at c, but in a strange way

Now let:

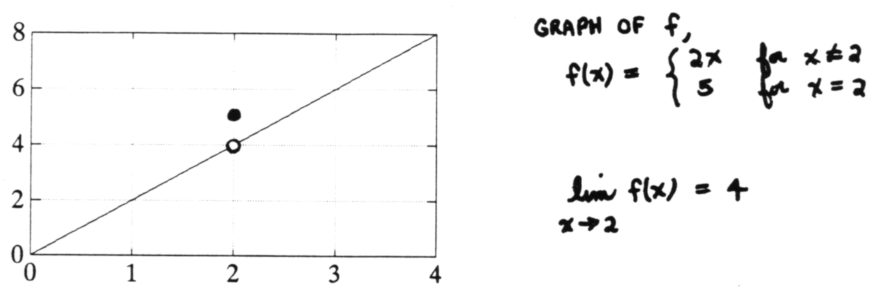
$$f(x) = \begin{cases} 2x & \text{for } x \ne 2 \\ 5 & \text{for } x = 2 \end{cases}$$

The graph of f is shown below. Again:

$$\lim_{x \to 2} f(x) = 4$$

This is because when x is close to 2 (but not equal to 2), the corresponding function values are close to 4.

When evaluating a limit as x approaches c, *x is not allowed to equal c*; the x values merely get *arbitrarily close to c*.



EXERCISE 3	♣ 1. Sketch the graph of a function that satisfies the following conditions: • $\lim_{x \to 3} f(x) = 4$ • $f(3) = 2$ ♣ 2. Sketch the graph of a function that satisfies the following conditions: • $\lim_{x \to 0} f(x) = 1$ • $0 \notin \mathcal{D}(f)$

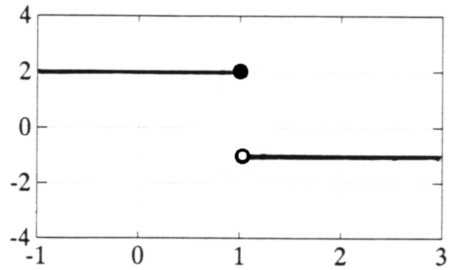
EXAMPLE

a limit that does not exist

Let:

$$f(x) = \begin{cases} 2 & x \le 1 \\ -1 & x > 1 \end{cases}$$

In this case, $\lim_{x\to 1} f(x)$ does not exist. The two required conditions cannot possibly be met for *any* real number l. As x gets close to 1 from the left-hand side, the function values are all equal to 2. As x gets close to 1 from the right-hand side, the function values are all -1. Thus, we are *not* getting close to *the same number* from both sides.



GRAPH OF f

$\lim_{x\to 1} f(x)$ DOES NOT EXIST

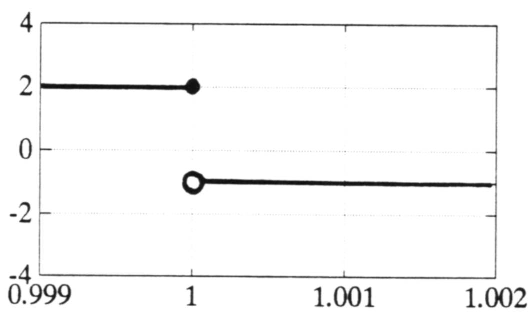
EXAMPLE

Let's work with the same function as in the previous example, but now consider some values of c different from 1.

What is $\lim_{x\to 2} f(x)$? As x approaches 2 from the right and left sides, $f(x)$ is -1. Thus, $\lim_{x\to 2} f(x) = -1$.

Similarly, $\lim_{x\to 0} f(x) = 2$.

What about $\lim_{x\to 1.001} f(x)$? When x is (sufficiently) close to 1.001, what (if anything) are the corresponding outputs close to? To answer this question, we need only 'magnify' what's happening for values of x near 1.001, as in the graph below. Now, it's clear that $\lim_{x\to 1.001} f(x) = -1$.



A MAGNIFICATION OF WHAT'S HAPPENING NEAR $x = 1.001$

$\lim_{x\to 1.001} f(x) = -1$

EXERCISE 4

Consider the function g given by:

$$g(x) = \begin{cases} -3 & \text{for } x < 2 \\ 5 & \text{for } x \geq 2 \end{cases}$$

♣ 1. Sketch the graph of g.

♣ 2. Find the following numbers, if they exist. Be sure to write complete mathematical sentences.

a) $g(2)$

b) $g(1.9782)$

c) $g(\pi)$

d) $\lim_{x\to 2} g(x)$

e) $\lim_{x\to 3} g(x)$

f) $\lim_{x\to 1.99999} g(x)$

g) $\lim_{z\to 0} g(z)$ (Hint: z is a dummy variable.)

h) $\lim_{y\to \pi} g(y)$

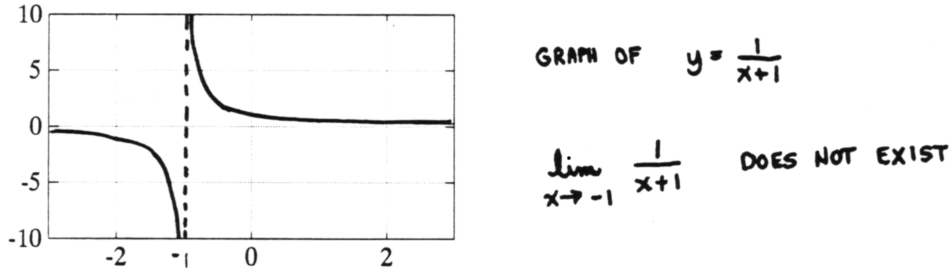
♣ 3. Let c be any number greater than 2. What is $\lim_{x\to c} g(x)$?

EXAMPLE
a limit that does not exist

The limit

$$\lim_{x\to -1} \frac{1}{x+1}$$

does not exist. As x approaches -1 from the left-hand side, $\frac{1}{x+1}$ approaches negative infinity. As x approaches -1 from the right-hand side, $\frac{1}{x+1}$ approaches positive infinity. The function values are not approaching any fixed real number.



The following sentences are all true:

$$\lim_{x\to 0} \frac{1}{x+1} = 1 \qquad \lim_{x\to 2} \frac{1}{x+1} = \frac{1}{3} \qquad \lim_{x\to -2} \frac{1}{x+1} = \frac{1}{-2+1} = -1$$

$$\lim_{y\to \pi} \frac{1}{y+1} = \frac{1}{\pi+1}$$

Also, for $c \neq -1$:

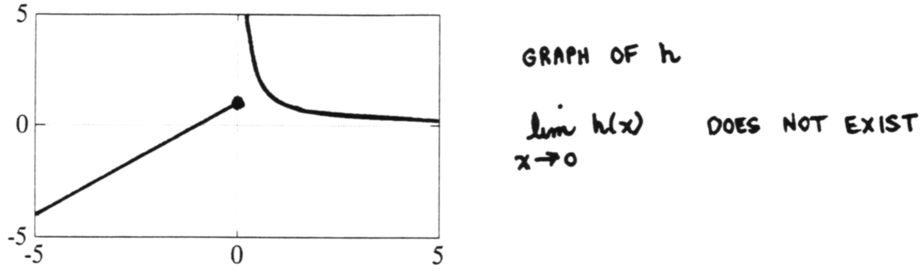
$$\lim_{x\to c} \frac{1}{x+1} = \frac{1}{c+1}$$

EXAMPLE

Let h be defined by:

$$h(x) = \begin{cases} \frac{1}{x} & \text{for } x > 0 \\ x + 1 & \text{for } x \le 0 \end{cases}$$

The graph of h is shown below.



As x approaches 0 from the left-hand side, $h(x)$ approaches 1. However, as x approaches 0 from the right-hand side, $f(x)$ does not approach 1. Thus, $\lim_{x\to 0} h(x)$ does not exist.

The following sentences are all true:

$$\lim_{x\to 2} h(x) = 1/2 \qquad \lim_{t\to -1} h(t) = 0 \qquad \lim_{x\to 10^{-5}} h(x) = 10^5$$

$$\lim_{x\to -10^{-5}} h(x) = 1 - 0.00001 = 0.99999$$

EXERCISE 5

Let f be defined by:

$$f(x) = \begin{cases} \frac{1}{x-2} & \text{for } x > 2 \\ 1 - x^2 & \text{for } x \le 2 \end{cases}$$

♣ 1. Sketch the graph of f.

♣ 2. What is the domain of f?

♣ 3. Find the following numbers, if they exist. Be sure to write complete mathematical sentences.

a) $f(2)$

b) $\lim_{x\to 2} f(x)$

c) $f(c)$, for $c > 100$

d) $f(t)$, for negative t

e) $\lim_{t\to \pi+1} f(t)$

f) $\lim_{\omega\to 0} f(\omega)$

Be sure you understand the difference between the *expression*

$$\lim_{x\to c} f(x) \tag{†}$$

and the *sentence*:

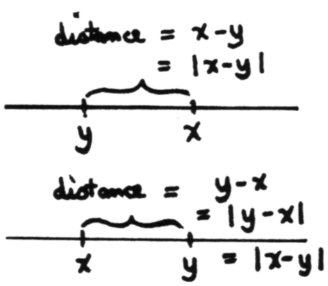
$$\lim_{x\to c} f(x) = l \tag{‡}$$

When (†) is defined, it is a NUMBER. What number? The number that $f(x)$ gets close to, as x gets close to c.

However, (‡) is a SENTENCE. Sentences have verbs; the verb in (‡) is the equals sign. This sentence (when it's true) is telling us that the number $\lim_{x\to c} f(x)$ is equal to l.

distance between real numbers

The use of the absolute value as a tool for measuring the distance between numbers is discussed next. This will help in understanding the precise definition of limit, which is the topic of the next section.



Let x and y be any two real numbers. Then:

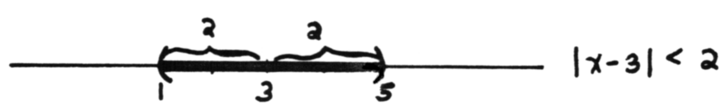
$$\text{the distance from } x \text{ to } y = |x - y|$$

Let's think about why this is true. If $x = y$, then the distance between them is 0, and the formula works.

If $x \neq y$, then one of the numbers lies further to the right on the number line. If x lies further to the right, the distance between the numbers is $x - y$. If y lies further to the right, the distance between the numbers is $y - x$. But in both cases, $|x - y|$ (which is equal to $|y - x|$) gives the distance between the two numbers.

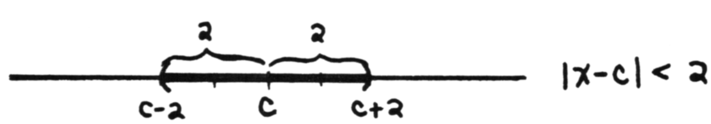
analyze the sentence $|x - 3| < 2$

Think about the sentence $|x - 3| < 2$. This sentence is an *inequality*; the verb is '$<$'. When is this sentence true? Using the interpretation of $|x - 3|$ as the distance between x and 3, the answer is easy: it is true for all numbers x whose distance from 3 is less than 2. Thus, it is true for $x \in (1, 5)$.



analyze the sentence $|x - c| < 2$

Now consider the sentence $|x - c| < 2$. By mathematical conventions, x is the variable, and c is a constant. When is this sentence true? Whenever x is a number whose distance from c is less than 2. Thus, the solution set of $|x - c| < 2$ is $(c - 2, c + 2)$.



analyze the sentence $0 < |x - c|$

For what values of x is $0 < |x - c|$ true? Reading from right to left, we must have the distance from x to c greater than 0. This happens as long as x is not equal to c; so the solution set of $0 < |x - c|$ is $(-\infty, c) \cup (c, \infty)$.

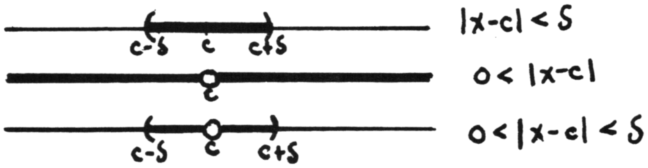
analyze the sentence $0 < |x - c| < \delta$

Now consider the sentence $0 < |x - c| < \delta$. By mathematical conventions, x is the variable, c and δ are constants. Usually, δ is thought of as a small positive number.

When is the sentence $0 < |x - c| < \delta$ true? Remember that this is short for *two* sentences, connected by the mathematical word 'and'. That is:

$$0 < |x - c| < \delta \iff 0 < |x - c| \text{ and } |x - c| < \delta$$

Thus, in order for the sentence to be true, we must have the distance from x to c less than δ, AND, x is not allowed to equal c. The solution set of $0 < |x - c| < \delta$ is shown below.



EXERCISE 6

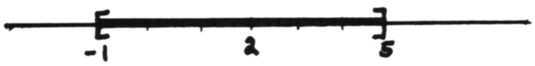
♣ 1. Write a mathematical sentence that is TRUE for all numbers whose distance from 4 is less than 2. What is the variable in your sentence?

♣ 2. Write a mathematical sentence that is TRUE for all numbers whose distance from -1 is greater than 5. What is the variable in your sentence?

♣ 3. Write a mathematical sentence that is TRUE for all numbers whose distance from π is greater than or equal to δ. (Here, δ is a constant.) What is the variable in your sentence?

EXERCISE 7

♣ 1. Write a mathematical sentence whose solution set is the set shown below.



♣ 2. Write a mathematical sentence whose solution set is the set shown below.

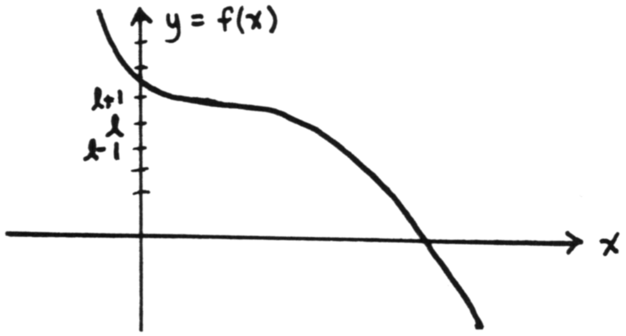
♣ 3. On a number line, show the solution set of $0 < |x - 3| < 5$.

♣ 4. On a number line, show the solution set of $0 < |x + 1| \le 2$. It may be helpful to rewrite $x + 1$ as $x - (-1)$.

EXERCISE 8

♣ The graph of a function f is shown below. A specific number l is labeled on the y-axis.

On the x-axis, clearly show $\{x \ : \ |f(x) - l| < 2\}$. Here, the colon ':' is used instead of the vertical bar, '|', so there is less confusion with the adjacent absolute value symbol. The colon ':' is still read as 'such that' or 'with the property that'.



alternate notation for limits

Suppose that the sentence

$$\lim_{x \to c} f(x) = l$$

is true. Then, as x approaches c, the numbers $f(x)$ must approach l. One often writes this as:

$$\text{As } x \to c, \ \ f(x) \to l \ .$$

This is read as: *As x approaches c, $f(x)$ approaches l.* Thus, the arrow '→' is read as 'approaches'.

QUICK QUIZ

sample questions

1. Evaluate the limit $\lim_{x \to -2} x^3$, if it exists.
2. Sketch the graph of $f(x) = x^2 \frac{x+1}{x+1}$. Then, evaluate the limit $\lim_{x \to -1} f(x)$, if it exists.
3. Sketch the graph of:

$$f(x) = \begin{cases} 3x & \text{for} \quad x \neq 1 \\ 5 & \text{for} \quad x = 1 \end{cases}$$

Then, evaluate the limit $\lim_{x \to 1} f(x)$, if it exists.

4. Sketch the graph of a function that satisfies the following conditions: $\lim_{x \to 2} f(x) = 5$, $f(2) = 1$.
5. Write a mathematical sentence that is TRUE for all numbers whose distance from -1 is less than or equal to 4. Use the variable t in your sentence.

KEYWORDS

for this section

Be familiar with the mathematical sentence:

$$\lim_{x \to c} f(x) = l$$

Roughly, when is this sentence true? Know that c and l are constants, and x is a dummy variable. Be able to evaluate simple limit statements. Know the difference between the expression $\lim_{x \to c} f(x)$ and the sentence $\lim_{x \to c} f(x) = l$. Know that the distance between real numbers x and y is given by $|x - y|$.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

1. $\lim_{x\to 1} 3x$
2. $\lim_{x\to 1} 3x = 3$
3. $\lim_{t\to 0} t^2 = 0$
4. $\lim_{t\to 0} t^2$
5. $\lim_{x\to c} f(x) = l$
6. As $x \to 1$, $2x \to 2$.
7. As $t \to 0$, $2t + 1 \to 1$.
8. $\lim_{x\to 2} f(x) = f(2)$
9. $\lim_{x\to 1} g(x) = g(1)$
10. $\lim_{x\to 0} \frac{x^2+x}{x} = 1$
11. $|x - y|$
12. $|x - 1| \le 2$
13. $|x - y| = |y - x|$
14. $|-2x - 2y| = 2|x + y|$
15. $|ab| = |a| \cdot |b|$
16. $|a + b| = |a| + |b|$
17. $|a - b| = |a| - |b|$
18. $|x| > 0 \iff x \in (-\infty, 0) \cup (0, \infty)$
19. For $\epsilon > 0$, $0 < |x| < \epsilon \iff x \in (-\epsilon, 0) \cup (0, \epsilon)$
20. For $0 < a < b$, $a < |x| < b \iff x \in (-b, -a)$ or $x \in (a, b)$