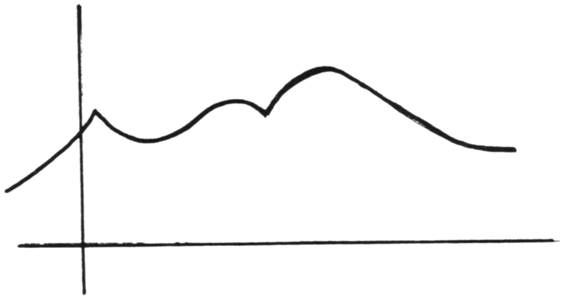
3.4 Continuity

Introduction

Intuitively, a function is *continuous* if its graph can be traced without lifting a pencil. This notion is made precise in this section.

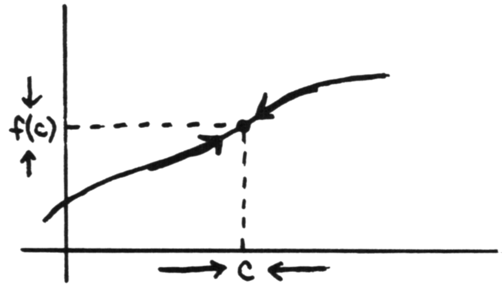


Mathematicians define what it means for a function to be *continuous at a point*: roughly, 'f *is continuous at the point* $(c, f(c))$' means that:

- f is defined at c (so that $f(c)$ makes sense)
- as x approaches c, $f(x)$ approaches $f(c)$

The phrase 'f *is continuous at the point* $(c, f(c))$' is usually shortened to 'f *is continuous at* c'.

The precise definition follows:



DEFINITION

continuity at a point

A function f is *continuous at* c if f is defined at c, and:

$$\lim_{x\to c} f(x) = f(c)$$

this definition is saying three things

It is important to realize that the statement $\lim_{x\to c} f(x) = f(c)$ is saying **three things**:

1) $f(c)$ exists (i.e., c is in the domain of f)
2) $\lim\limits_{x\to c} f(x)$ exists
3) $\lim\limits_{x\to c} f(x) = f(c)$ (that is, the numbers above are equal!)

f is discontinuous at c

If *any one* of these three criteria fail, then f is *not* continuous at c. In this case, one says that f is *discontinuous at c.*

Finding limits at a point of continuity is easy! Use direct substitution.

Suppose f is continuous at c. Then, we know that f is defined at c; that is, $f(c)$ exists. Also, evaluating $\lim_{x\to c} f(x)$ is as easy as direct substitution, since *continuity at c* tells us that the limit is equal to $f(c)$!

Example

In the previous section, it was shown that for any polynomial P:

$$\lim_{x\to c} P(x) = P(c)$$

Here, c is any real number. This result says that *polynomials are continuous everywhere.*

Thus, for example:

$$\lim_{x\to 7.2}(4x^4 - \sqrt{2}x + \pi) = 4(7.2)^4 - \sqrt{2}(7.2) + \pi$$

EXERCISE 1

♣ 1. Evaluate the following limits:

a) $\lim_{x\to 2}(x^2 - x + 1)$

b) $\lim_{x\to \pi}(x^2 - x + 1)$

c) $\lim_{x\to b}(x^2 - x + 1)$ (here, b is a real number)

d) $\lim_{x\to n}(x^2 - x + 1)$ (here, n is an integer)

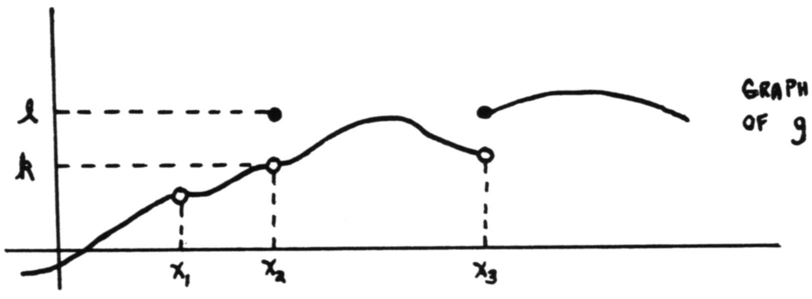
e) $\lim_{x\to d}(ax^2 + bx + c)$ (here, a, b, c and d are real numbers)

♣ 2. Find a function f and a number c such that f is continuous at c and $\lim_{x\to c} f(x) = 2$.

How can a function FAIL to be continuous at c?

Since there are really three requirements for a function to be continuous at c, there are also three ways that a function can *fail* to be continuous at c.

The function g with graph shown below illustrates the three ways that a function can be discontinuous at c. Refer to this graph for the discussions below.



f may not be defined at c

Whenever a function is not defined at c, (that is, $f(c)$ does not exist), then f is not continuous at c.

The function g is discontinuous at x_1, because g is not defined at x_1.

the limit as x approaches c may not exist

If $\lim_{x\to c} f(x)$ does not exist, then f is not continuous at c.

This function g is discontinuous at x_3, because $\lim_{x\to x_3} g(x)$ does not exist.

both $f(c)$ and $\lim\limits_{x\to c} f(x)$ may exist, but they aren't equal

It is possible for both $f(c)$ and $\lim_{x\to c} f(x)$ to exist, but not be equal.

The function g is discontinuous at x_2.

In this case, g is defined at x_2; $g(x_2) = l$.

Also, the limit of g as x approaches x_2 exists; $\lim_{x\to x_2} g(x) = k$.

However, these two numbers are *not equal*! That is:

$$\lim_{x\to x_2} g(x) \neq g(x_2)$$

two types of discontinuities

If a function is discontinuous at c, then the discontinuity can be classified, depending on how the definition of continuity fails.

DEFINITION
removable discontinuity

A function f has a *removable discontinuity at c* whenever $\lim_{x\to c} f(x)$ exists, but is not equal to $f(c)$.

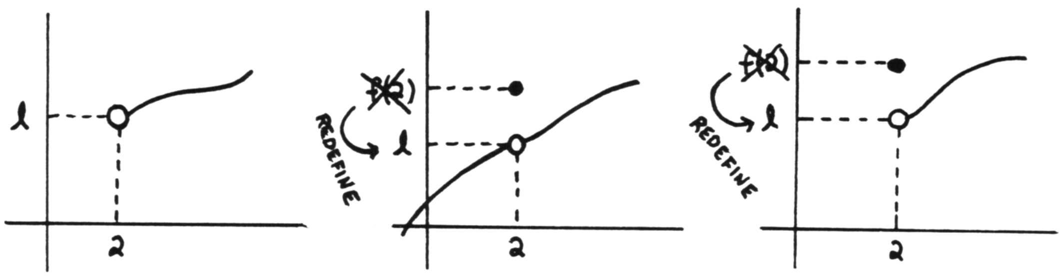
In this case, the discontinuity can be easily *removed* by merely defining (or redefining) the function f at c!

EXAMPLE
removable discontinuities

The functions whose graphs are shown below all have removable discontinuities at $x = 2$.

In the first case, $\lim_{x\to 2} f(x)$ exists, but f is not defined at 2. This discontinuity can be easily removed by defining $f(2) = l$.

In the second and third cases, $\lim_{x\to 2} f(x)$ exists, and $f(2)$ exists, but these numbers are not equal. These discontinuities can be easily removed by redefining f at 2 so that $f(2) = l$.



DEFINITION
nonremovable discontinuity

A function f has a *nonremovable discontinuity at c* whenever $\lim_{x\to c} f(x)$ does not exist.

In this case, the discontinuity can *not* be easily removed!

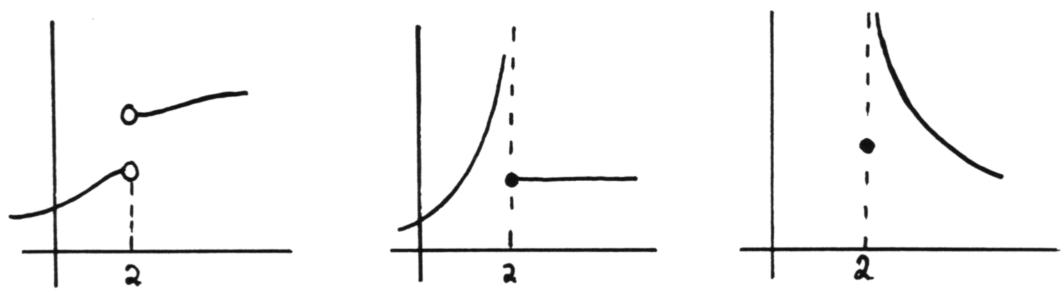
EXAMPLE

nonremovable discontinuities

The functions with graphs shown below all have nonremovable discontinuities at $x = 2$. In all cases, $\lim_{x\to 2} f(x)$ does not exist.

Observe that f may or may not be defined at a nonremovable discontinuity.

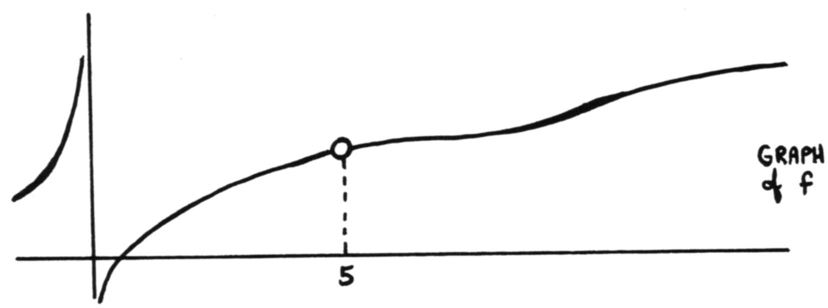
Any attempt to 'patch up' these discontinuities would require major reconstructive work. Essentially, one must grab both pieces of the graph and pull them together. No matter how this is done, it requires redefining the function on some entire *interval*, as opposed to just at a single point. Thus, this type of discontinuity is *not* easy to remove!



classifying discontinuities

To *classify a discontinuity* means to state if the discontinuity is removable or nonremovable.

For example, suppose you are asked to classify the discontinuities of the function shown below:



The correct response is:

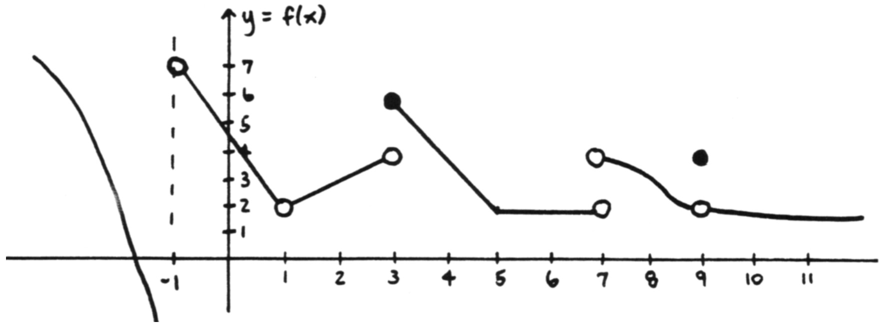
f has a removable discontinuity at 5.

f has a nonremovable discontinuity at 0.

Be sure to write complete mathematical sentences! Do *not* merely say: '*removable discontinuity at* 5'.

EXERCISE 2

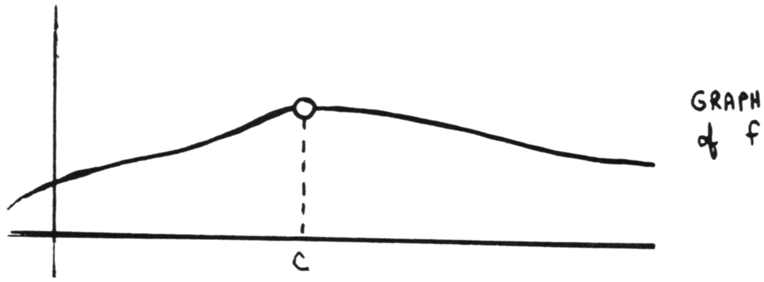
♣ 1. Classify the discontinuities of the function shown below.



♣ 2. If a discontinuity is removable, indicate how it can be 'removed'.

DON'T ASK THIS QUESTION!

Consider the function f shown below. Students like to ask the question: *Is this function continuous?*



Now, does this question really make sense? Continuity has only been defined *at a point*! That is, we have not defined what it means for a function f *to be continuous*; we have only defined what it means for a function f *to be continuous at a point* c.

The function shown is not continuous at c, because it's not defined at c. But, f IS continuous at every point where it is defined. Therefore, the absolutely correct answer to the not-so-correct question *Is this function continuous?* is:

The function f is continuous at every point in its domain.

or,

The function f is continuous at every point where it is defined.

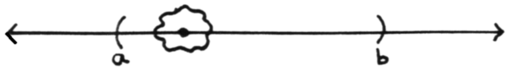
continuity on an interval

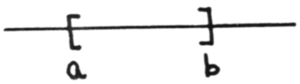
If a function happens to be continuous on an entire *interval* of real numbers, then it has some particularly nice properties. First, a brief discussion of *intervals*.

open intervals

closed intervals

A finite interval is said to be *open* if it does not include either endpoint. Thus, (a, b) is an open interval. The word 'open' refers to the fact that *every* point in this interval *has some space around it (on both sides) that remains entirely inside the interval.* In other words, each point in the interval has some room both to the left and to the right that is still in the interval. (Think of 'the wide open spaces'!)





A finite interval is said to be *closed* if it includes both endpoints. Thus, $[a, b]$ is a closed interval. Observe that the endpoints a and b do NOT have room both to the right and left that is still in the interval. There is no room to the left of a; and there is no room to the right of b.

A finite interval that includes only one endpoint is not open and not closed. Thus, the intervals $(a, b]$ and $[a, b)$ are not open, and not closed. *Thus, the words 'open' and 'closed' are used differently in mathematics than in English.* In English, if a door is not open, then it is closed. In mathematics, just because an interval is not open, does NOT mean that it is closed.

DEFINITION

continuity on an interval $[a, b]$

A function f is *continuous on the interval* $[a, b]$ if it satisfies the following conditions:

- f is defined on $[a, b]$
- f is continuous at each point in (a, b)
- As x approaches a from within the interval (from the right), $f(x)$ approaches $f(a)$. That is, f is well-behaved at the left endpoint.

  This can be stated in terms of a right-hand limit:

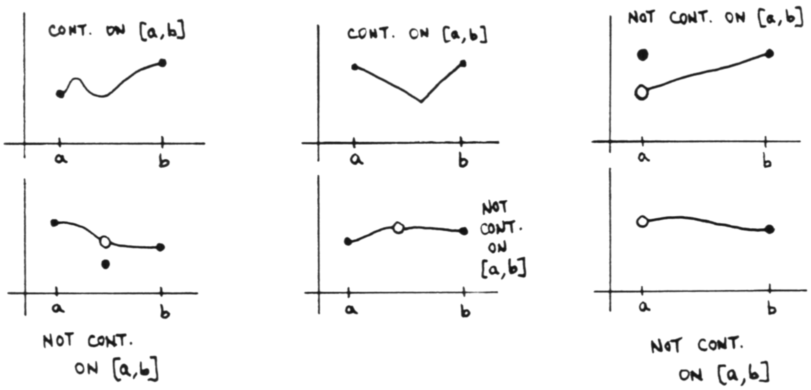
  $$\lim_{x \to a^+} f(x) = f(a)$$

- As x approaches b from within the interval (from the left), $f(x)$ approaches $f(b)$. That is, f is well-behaved at the right endpoint.

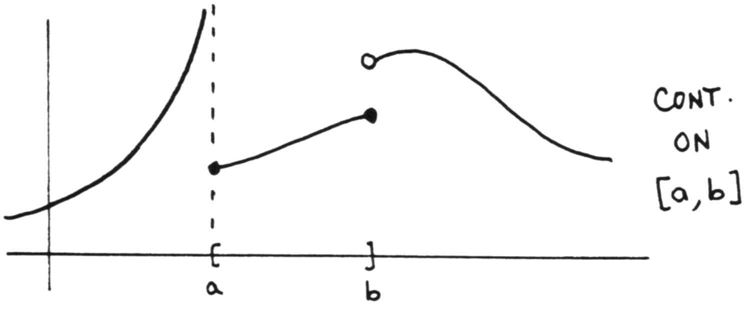
  This can be stated in terms of a left-hand limit:

  $$\lim_{x \to b^-} f(x) = f(b)$$

Some illustrative sketches appear below.



NOTE: A function f may be defined outside of $[a,b]$. However, when answering the question, '*Is f continuous on $[a,b]$?*', the function outside of $[a,b]$ is IGNORED. That is, to investigate the continuity of a function f on $[a,b]$, all one cares about is how f acts on $[a,b]$, and not outside of this interval.



A function that is continuous on a closed interval satisfies some particularly nice properties. These properties will be investigated in the last two sections of this chapter.

EXERCISE 3

Sketch graphs of functions satisfying the following requirements:

♣ 1. f is defined on $[a,b]$, but $\lim_{x\to a^+} f(x) \neq f(a)$

♣ 2. f is defined on $[a,b]$, but $\lim_{x\to b^-} f(x) \neq f(b)$

♣ 3. f is continuous on $[a,b]$, $f(a) = 2$ and $f(b) = 0$

♣ 4. f is defined on all of $\mathbb{R}$, f is continuous on $[a,b]$, but f is not continuous at a

continuity of sums, products, and scalar multiples

Suppose that functions f and g are both continuous at c. That is:

$$\lim_{x\to c} f(x) = f(c) \quad \text{and} \quad \lim_{x\to c} g(x) = g(c)$$

Then, by properties of limits:

$$\lim_{x\to c} f(x) + g(x) = f(c) + g(c)$$
$$\lim_{x\to c} f(x) \cdot g(x) = f(c) \cdot g(c)$$
$$\lim_{x\to c} k \cdot f(x) = k \cdot f(c)$$

Thus, if f and g are both continuous at c, then so are the sum $f+g$, the product $f \cdot g$, and the scalar multiple kf.

EXERCISE 4

quotients of continuous functions

♣ State a similar result regarding *quotients* of functions that are both continuous at c.

continuity of composite functions

Under what conditions should the composition $(f \circ g)$ be continuous at c? When x is close to c, we want $f(g(x))$ close to $f(g(c))$.

This can be guaranteed in two steps.

First, require that when x is close to c, then $g(x)$ is close to $g(c)$. That is, require that g be continuous at c.

Next, require that when the inputs to f are close to the number $g(c)$, then the outputs are close to $f(g(c))$. That is, require that f be continuous at $g(c)$.

Precisely, if g is continuous at c and f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

EXERCISE 5

Suppose that $g(3) = 9$ and $f(9) = 2$.

♣ 1. Under what conditions on f and g will the composition $f \circ g$ be continuous at 3?

♣ 2. Under these conditions, what is

$$\lim_{x \to 3} f(g(x)) \ ?$$

QUICK QUIZ

sample questions

1. Give a precise definition of what it means for a function f to be continuous at c.
2. Suppose that $\lim_{x \to c} f(x) = 2$ and $f(c) = 3$. Is f continuous at c? If not, classify the discontinuity.
3. Under what condition(s) does f have a nonremovable discontinuity at c?
4. For a given function f and constant $c \in \mathcal{D}(f)$, under what condition(s) is evaluating the limit $\lim_{x \to c} f(x)$ as easy as 'direct substitution'?
5. Sketch the graph of a function satisfying the following properties: $\mathcal{D}(f) = [1, 3]$, $f(1) = 2$, f is NOT continuous at $x = 1$, f IS continuous at $x = 3$.

KEYWORDS

for this section

Precise definition of continuity at a point: What three things is this definition saying? When can direct substitution be used to find a limit? Removable and nonremovable discontinuities, classifying discontinuities, open and closed intervals, difference between English and mathematical usage of the words 'open' and 'closed', continuity on an interval $[a, b]$, continuity of sums, products, scalar multiplies, quotients, and composite functions.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

1. f is continuous at c
2. $f(c) = 5$
3. $f(c)$
4. $\lim_{x\to c} f(x)$
5. $\lim_{x\to c} f(x) = f(c)$
6. If P is a polynomial, then $\lim_{x\to c} P(x) = P(c)$.
7. The function f has a removable discontinuity at $x = 2$.
8. If $\lim_{x\to c} f(x)$ does not exist, then f has a nonremovable discontinuity at c.
9. The function f is continuous on $[a, b]$.
10. The function $f(x) = x^2$ is continuous on $[1, 3]$.
11. If functions f and g are both continuous at c, then so is $f + g$.
12. If functions f and g are both continuous at $x = 2$, then so is the product function fg.
13. If a finite interval of real numbers is not open, then it is closed.
14. If a finite interval of real numbers is not closed, then it is open.
15. (a, b)
16. $(a, b]$ is an open interval