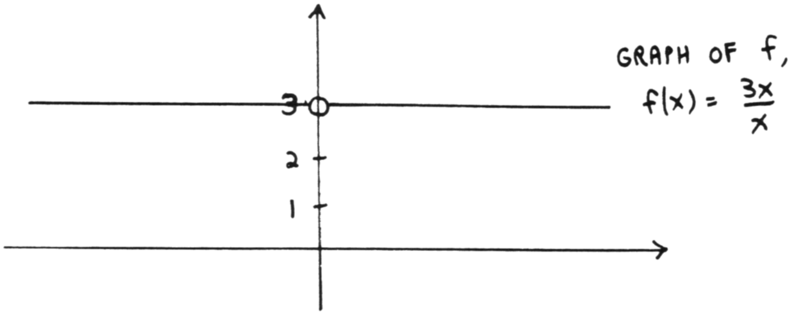
3.5 Indeterminate Forms

Introduction;
$\frac{0}{0}$ situations

Consider the function given by the rule $f(x) = \frac{3x}{x}$; its graph is shown below. Clearly, $\lim_{x\to 0} \frac{3x}{x} = 3$. Note, however, that *if one merely tried to plug in* 0 *for* x when investigating this limit, a '$\frac{0}{0}$' situation would have occurred.



Whenever direct substitution into $\lim_{x\to c} f(x)$ yields a $\frac{0}{0}$ situation, then the function f is *not* defined at c, since division by zero is not allowed. *But the limit MAY still exist.* Or, it may not exist. To see which of these two situations occurs, it is necessary to *rewrite* the function f to get it into a form where one can better analyze what's happening **near** c.

EXAMPLE
a $\frac{0}{0}$ situation;
the limit exists

Problem: Evaluate the limit $\lim_{x\to 0} \frac{3x}{x}$.

Solution: Remember that when investigating a limit as x approaches 0, x is not allowed to equal 0. And, for all values of x except 0, $\frac{3x}{x} = 3$. Therefore, the limit statement can be rewritten in an easier form:

$$\begin{aligned} \lim_{x\to 0} \frac{3x}{x} &= \lim_{x\to 0} 3 && (\frac{3x}{x} = 3 \text{ whenever } x \neq 0) \\ &= 3 && (\text{the limit of a constant function}) \end{aligned}$$

When evaluating the limit $\lim_{x\to c} f(x)$*, the function* f *may be replaced by ANY function that agrees with* f *NEAR* c *(but not necessarily AT* c*).*

EXERCISE 1

♣ Evaluate the limit:

$$\lim_{x\to 1} \frac{3-3x}{x-1}$$

Be sure to write complete mathematical sentences.

EXAMPLE
a $\frac{0}{0}$ situation;
the limit does not exist

Problem: Evaluate the limit:

$$\lim_{x\to 0} \frac{3x}{x^2}$$

Solution: Again, direct substitution yields a $\frac{0}{0}$ situation.

$$\lim_{x\to 0} \frac{3x}{x^2} = \lim_{x\to 0} \frac{3}{x}$$

Since $\lim_{x\to 0} \frac{3}{x}$ does not exist, neither does $\lim_{x\to 0} \frac{3x}{x^2}$.

EXAMPLE

Problem: Evaluate the limit:

$$\lim_{x\to 1} \frac{x^2+x-2}{x-1}$$

Solution: Direct substitution of $x = 1$ into this limit statement yields a $\frac{0}{0}$ situation. It is necessary to rewrite $\frac{x^2+x-2}{x-1}$ in a way that better displays what is happening *near* $x = 1$. Since 1 is a zero of the numerator, $x - 1$ is a factor of the numerator. Indeed, factoring and canceling yields:

$$\begin{aligned} \frac{x^2+x-2}{x-1} &= \frac{(x-1)(x+2)}{x-1} && \text{(factor the numerator)} \\ &\overset{\text{for } x\neq 1}{=} x+2 && \text{(cancel factor of 1)} \qquad (*) \end{aligned}$$

Thus:

$$\begin{aligned} \lim_{x\to 1} \frac{x^2+x-2}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x+2)}{x-1} \\ &= \lim_{x\to 1} (x+2) \\ &= 3 \end{aligned}$$

a 'restricted' equal sign

The second '=' sign that appears in (*) above is a sort of 'restricted' equal sign.

It is NOT completely correct to say that $\frac{(x-1)(x+2)}{x-1} = x + 2$, since these two expressions are NOT equal for all values of x. The left-most expression is not defined when x is 1; the right-most expression is 3 when x is 1.

To bring attention to this difference in the expressions, the necessary restriction is indicated over the equal sign. Thus, the reader becomes aware that the equality only holds when x is not equal to 1.

If the 'restricted' equal sign seem bothersome to you, there is an alternate (but more cumbersome) solution. One can instead say:

$$\frac{x^2+x-2}{x-1} = \frac{(x-1)(x+2)}{x-1} \ ;$$

and for $x \neq 1$, this latter expression simplifies to $x + 2$.

an important distinction

The equal sign in the sentence

$$\lim_{x\to 1} \frac{(x-1)(x+2)}{x-1} = \lim_{x\to 1} (x+2)$$

is *not* a restricted equal sign. This equal sign is asserting that two *real numbers* are equal. Remember that every limit, if it exists, is a real number.

However, take away the limit instruction, and a restricted equal sign *is* needed:

$$\frac{(x-1)(x+2)}{x-1} \overset{\text{for } x\neq 1}{=} x+2$$

One way to view the '=' sign in this latter sentence, is that it is being used to compare two *functions*. Two *functions* are equal only if they have the same domains, and the outputs agree for all allowable inputs. A precise statement follows.

DEFINITION
equality of functions

Let f and g be functions of one variable. Then:

$$f = g \quad \iff \quad \begin{array}{l}\mathcal{D}(f) = \mathcal{D}(g) \text{ and} \\ f(x) = g(x) \text{ for all } x \text{ in} \\ \text{the common domain}\end{array}$$

EXERCISE 2

♣ 1. What mathematical sentence is being defined in the previous definition?

♣ 2. What does the symbol ‘ $\iff$ ’ mean in this definition?

♣ 3. Suppose you are told that g and h are both functions of one variable, and $g = h$. What can you conclude?

♣ 4. Suppose you are told that g and h are both functions of one variable, $\mathcal{D}(g) = \mathcal{D}(h) := \mathcal{D}$, and $g(x) = h(x) \ \ \forall\, x \in \mathcal{D}$. What can you conclude?

♣ 5. Let f and g be defined by the rules:

$$f(x) = x \cdot \frac{x-1}{x-1} \quad \text{and} \quad g(x) = x$$

Does $f = g$?

♣ 6. Let f and g be defined by the rules:

$$f(x) = \frac{3x}{x^2} \quad \text{and} \quad g(x) = \frac{3}{x}$$

Does $f = g$?

EXERCISE 3

Evaluate the following limits. Be sure to write complete mathematical sentences.

♣ 1. $\displaystyle\lim_{x\to 1} \frac{x^2 + x - 2}{3x^2 - x - 2}$

♣ 2. $\displaystyle\lim_{x\to -2} \frac{x^2 + x - 2}{x^3 - 7x - 6}$

EXERCISE 4

In the sentences below, replace the question marks with an equal sign ‘=’ or an appropriate ‘restricted’ equal sign.

♣ 1. $\displaystyle\lim_{x\to 2} \frac{e^x(x-2)}{2-x} \quad ? \quad -\lim_{x\to 2} e^x \quad ? \quad e^2$

♣ 2. $\frac{xe^x - 2e^x}{2-x} \quad ? \quad -e^x$

EXAMPLE
a $\frac{\infty}{\infty}$ situation

Problem: Evaluate the limit:

$$\lim_{x\to 0} \frac{3/x}{1/x}$$

Solution: Direct substitution yields an $\frac{\infty}{\infty}$ situation. However, for $x \ne 0$ it is true that $\frac{3/x}{1/x} = 3$, and so:

$$\lim_{x\to 0} \frac{3/x}{1/x} = \lim_{x\to 0} 3 = 3$$

EXAMPLE

Problem: Evaluate the limit:

$$\lim_{x\to 0} \frac{1/x^2}{1/x}$$

Solution:

$$\lim_{x\to 0} \frac{1/x^2}{1/x} = \lim_{x\to 0} \frac{1}{x}$$

Since $\lim_{x\to 0} \frac{1}{x}$ does not exist, neither does $\lim_{x\to 0} \frac{1/x^2}{1/x}$.
Hence, limits of the form $\frac{\infty}{\infty}$ may or may not exist.

EXAMPLE

a 1^∞ situation

Consider next the one-sided limit:

$$\lim_{x\to 0^+} (1+x)^{1/x}$$

The chart below suggests what happens as x approaches zero from the positive side:

x	$(1+x)^{1/x}$
0.10000000000000	2.59374246010000
0.01000000000000	2.70481382942153
0.00100000000000	2.71692393223559
0.00010000000000	2.71814592682493
0.00001000000000	2.71826823719230
0.00000100000000	2.71828046909575
0.00000010000000	2.71828169413208
0.00000001000000	2.71828179834736

Based on these results, it should not be surprising that:

$$\lim_{x\to 0^+} (1+x)^{1/x} = e$$

(This limit is sometimes taken as the *definition* of the irrational number e.)

indeterminate forms;
$\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, 1^∞

Limits that result in a $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ or 1^∞ situation under direct substitution are called *indeterminate forms*. Such limits *may* or *may not* exist. They always require further analysis, to see what's really happening near the x-value of interest.

EXERCISE 5

Evaluate the following limits, if they exist:

♣ 1. $\lim_{x\to 1} \frac{x^3-1}{x-1}$

♣ 2. $\lim_{x\to 2} \frac{x^3-4x^2-11x+30}{x-2}$

evaluating a limit by rationalizing the numerator

Recall that *rationalizing* means to rewrite in a form with *no radicals.* Thus, to *rationalize the numerator* means to rewrite the numerator in a form that contains no radicals.

Consider $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$. Observe that direct substitution would result in a '$\frac{0}{0}$' situation. To get better insight into this limit, you might want to plug numbers like $-.001$ and $.001$ into $\frac{\sqrt{x+1}-1}{x}$. Does it appear to be getting close to any particular number?

Here's how to "massage" the function into a form that is more suitable for seeing what happens for values *near* zero:

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\\ &= \lim_{x\to 0}\frac{(x+1)-1}{x(\sqrt{x+1}+1)} = \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}\\ &= \frac{1}{2}\end{aligned}$$

Do you believe this answer, based on your earlier analysis?

EXERCISE 6

Let's investigate the previous example a bit more closely.

♣ 1. Is the sentence

$$\frac{(x+1)-1}{x(\sqrt{x+1}+1)} = \frac{1}{\sqrt{x+1}+1}$$

true? Why or why not? Is a 'restricted' equal sign needed here?

♣ 2. Is the sentence

$$\lim_{x\to 0}\frac{(x+1)-1}{x(\sqrt{x+1}+1)} = \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}$$

true? Why or why not? Is a 'restricted' equal sign needed here?

EXERCISE 7

♣ Evaluate $\lim_{x\to 0}\dfrac{3x}{\sqrt{x+4}-2}$. If the limit does not exist, so state.

QUICK QUIZ

sample questions

1. What is an 'indeterminate form'? Answer in a complete sentence.
2. Evaluate $\lim_{x\to 1}\frac{x^2-1}{x-1}$. If the limit does not exist, so state. Be sure to write a complete mathematical sentence.
3. Is the sentence
$$\frac{x^2-1}{x-1} = x+1$$
true for ALL values of x? Why or why not?
4. Graph the equation $y = \frac{x^2-1}{x-1}$.
5. Graph the function $f(x) = \frac{x^2-1}{x-1}$.
6. Let f and g be functions of one variable. Give a precise definition of the sentence '$f = g$'.

KEYWORDS *for this section*

What is meant by $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$ and 1^{∞} situations? Restricted equal sign, equality of functions, indeterminate forms, rationalizing the numerator.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

1. For all real numbers x, $\frac{x^3-1}{x-1} = x^2 + x + 1$.
2. For all real numbers x except 1, $\frac{x^3-1}{x-1} = x^2 + x + 1$.
3. Let $f(x) = \frac{x^3-1}{x-1}$ and $g(x) = x^2 + x + 1$. Then, $f = g$.
4. $\lim_{x\to 1} \frac{x^3-1}{x-1}$
5. $\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1}(x^2 + x + 1)$
6. When evaluating a limit $\lim_{x\to c} f(x)$, the function f can be replaced by any function g that agrees with f, except possibly at c.
7. Suppose that whenever $x \ne c$, $f(x) = g(x)$. Then, $\displaystyle\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$.
8. $f = g$

♣ Evaluate the following limits. If a limit does not exist, so state. Be sure to write complete mathematical sentences.

9. $\displaystyle\lim_{x\to -1} \frac{x^3 + x^2 - 3x - 3}{x+1}$
10. $\displaystyle\lim_{x\to 1} \frac{x^3 + x^2 - 3x - 3}{x+1}$
11. $\displaystyle\lim_{x\to 2} \frac{x+2}{x^2 + 4x + 4}$
12. $\displaystyle\lim_{x\to -2} \frac{x+2}{x^2 + 4x + 4}$
13. $\displaystyle\lim_{t\to 0^+} (1+t)^{1/t}$
14. $\displaystyle\lim_{y\to 0^+} (y+1)^{1/y}$