4.1 Tangent Lines

Introduction

Recall that the *slope of a line* tells us how fast the line rises or falls. Given distinct points (x_1, y_1) and (x_2, y_2), the slope of the line through these two points is

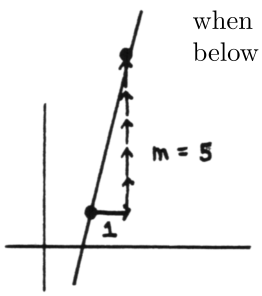
$$\frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} \ ,$$

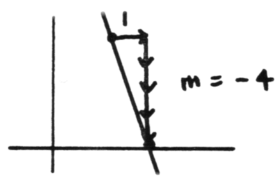
providing that $x_2 \neq x_1$. If $x_2 = x_1$, the line is vertical, and the slope *does not exist.*

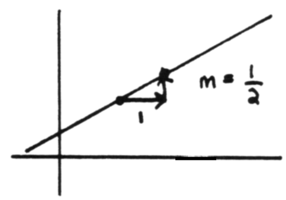
For given points (x_1, y_1) and (x_2, y_2) satisfying the additional requirement that $x_2 - x_1 = 1$, the slope of the line becomes:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{1}$$

This simple observation gives an important interpretation of the slope of a line: it is a number that tells the *vertical change per (positive) unit horizontal change* when traveling from point to point on the line. For example, the lines shown below have (from left to right) slopes 5, -4, and $\frac{1}{2}$.

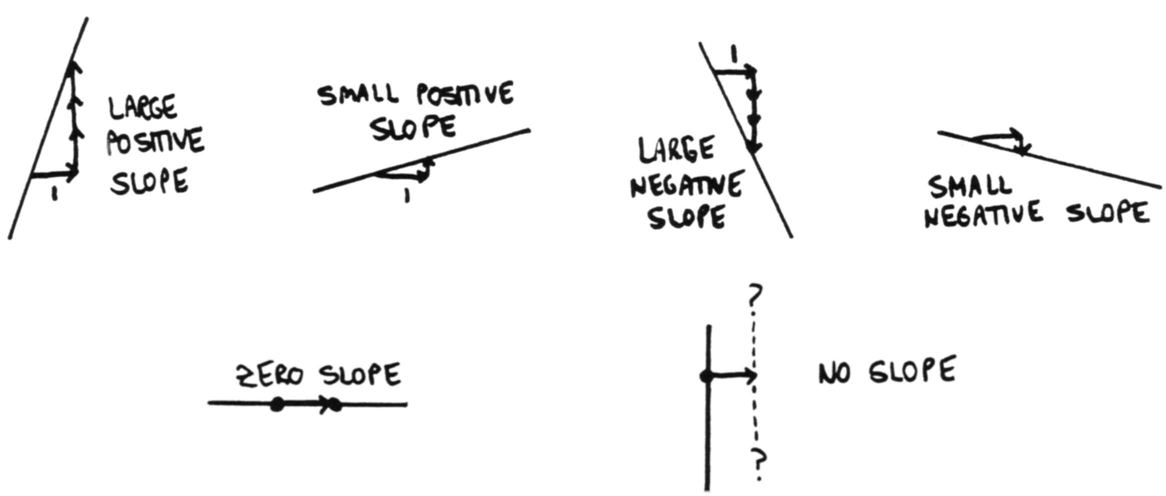






When traveling along a line from left to right:

- lines with large positive slopes are steep 'uphills';
- lines with small positive slopes are gradual 'uphills';
- lines with large negative slopes are steep 'downhills'; and
- lines with small negative slopes are gradual 'downhills'.



EXERCISE 1

♣ 1. Prove that:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

Therefore, the order that the points are listed when calculating the slope of a line is unimportant.

♣ 2. A line has slope 3. If the x-values of two points on the line differ by 1, how much do their y-values differ by? If the x-values of two points differ by 2, how much do their y-values differ by?

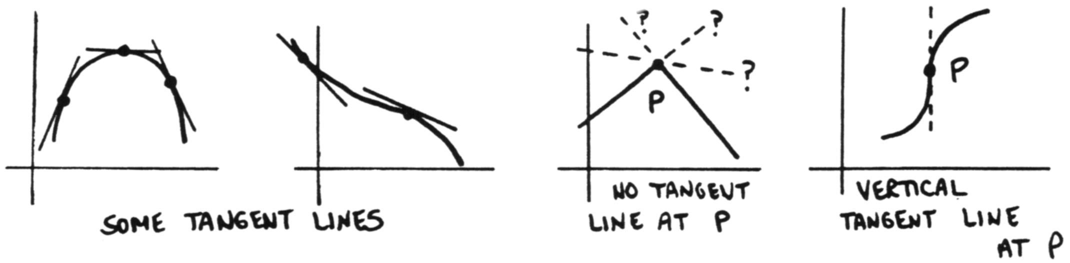
♣ 3. On the same graph, sketch lines that have slopes 1, 10, and $\frac{1}{10}$.

♣ 4. On the same graph, sketch lines that have slopes -1, -10, and $-\frac{1}{10}$.

tangent lines; informal discussion

The *tangent line* to a graph at a point P is the *line that best approximates the graph at that point.* In other words, it is the *best linear approximation at P.*

Tangent lines *may* or *may not* exist, as illustrated below. When they *do* exist, it is intuitively clear how they should be drawn.



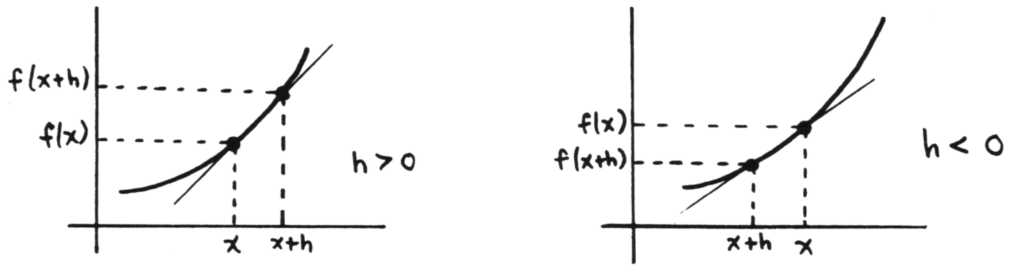
finding the slopes of tangent lines

GOAL: Find the slope of the tangent line to the graph of a function f at the point $(x, f(x))$.

PROBLEM: *Two* points are needed to find the slope of a line!

To remedy this problem, choose a second point that is close to $(x, f(x))$, and find the slope of the line through these two points. When the second point is very close to $(x, f(x))$, this line should be a good approximation to the tangent line.

Let h denote some small number, positive or negative. (Think of h as being, say, 0.1, 0.001 or -0.01.) Then, the point $(x+h, f(x+h))$ is close to $(x, f(x))$. If $h > 0$, the new point is to the right of $(x, f(x))$. If $h < 0$, the new point is to the left of $(x, f(x))$.



secant line

The line through these two points $(x, f(x))$ and $(x+h, f(x+h))$ is called a *secant line*. It serves as an approximation to the desired tangent line. In general, the *closer* the second point $(x+h, f(x+h))$ is to the initial point $(x, f(x))$, the better the approximation.

The slope of the secant line through the points $(x, f(x))$ and $(x+h, f(x+h))$ is:

$$\begin{aligned}\text{slope of secant line} &= \frac{f(x+h)-f(x)}{(x+h)-x} \\ &= \frac{f(x+h)-f(x)}{h}\end{aligned}$$

difference quotient

The quantity

$$\frac{f(x+h)-f(x)}{h}$$

obtained above is called a *difference quotient*. It represents the slope of the secant line through the points $(x, f(x))$ and $(x+h, f(x+h))$.

let $h \to 0$

Since we expect the slope of the secant line to better approximate the slope of the tangent line as the second point moves closer to the first (which happens as h approaches 0), it is natural to investigate the limit:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

This limit *may* or *may not* exist. If it *does* exist, then there *is* a tangent line to the graph of f at the point $(x, f(x))$, and the limit value gives the *slope of the tangent line to the graph of f at the point $(x, f(x))$.* This result is summarized next.

DEFINITION

slope of the tangent line to the graph of f at the point $(x, f(x))$.

If the limit

$$m = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists, then there is a nonvertical tangent line to the graph of f at the point $(x, f(x))$, and the number m gives the slope of this tangent line.

investigating the limit; what are x and h?

The limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

uses two letters, x and h. The letter h is the *dummy variable* for the limit; it merely represents a number that is getting arbitrarily close to zero. The limit can equally well be written with a different dummy variable, say:

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \text{or} \quad \lim_{t\to 0} \frac{f(x+t)-f(x)}{t}$$

(The symbol Δx is read as '*delta x*', and denotes a *change in x*.)

The letter x that appears in the limit is the x-value of the point where the slope of the tangent line is desired. If, for example, the slope is desired at the point $(2, f(2))$, then the limit becomes:

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

Note that the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ can only be investigated at a value of x where f is defined, so that $f(x)$ makes sense.

the limit is a $\frac{0}{0}$ situation

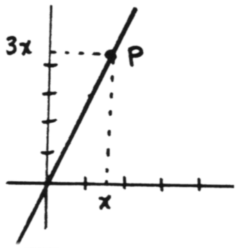
Observe that direct substitution of $h = 0$ into the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

yields a $\frac{0}{0}$ situation. Therefore, this limit can *never* be evaluated directly. It is necessary to get $\frac{f(x+h)-f(x)}{h}$ into a form that displays what is happening when h is *close to* zero, but not equal to zero. In many cases, one tries to simplify the difference quotient to a point where there is a factor of h in the numerator, that can be cancelled with the h in the denominator.

EXAMPLE

using the limit formula to find the slopes of tangent lines



It's always best to test a new result in a situation where you already know the answer. So, let's work first with the function $f(x) = 3x$. The graph of f is a line of slope 3. If P is *any point* on this line, then the tangent line at P is the line itself, and we should find that the slope of the tangent line is 3. Let's see if the above formula bears this out.

Let the 'first point' be $(x, f(x)) = (x, 3x)$, and let the 'second point' be $(x + h, f(x+h)) = (x+h, 3(x+h))$. The slope of the secant line between these two points is

$$\frac{f(x+h)-f(x)}{h} = \frac{3(x+h)-3x}{h} = \frac{3h}{h}\ ,$$

and thus:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3h}{h} = 3$$

Thus, for *any* point $(x, f(x))$ on the graph, the slope of the tangent line is 3, as expected.

EXERCISE 2

♣ 1. Consider the function $f(x) = -3x$. Using the limit formula, find the slope of the tangent line at the point $(1, -3)$.

♣ 2. Consider the function $f(x) = -3x$. Using the limit formula, find the slope of the tangent line at a typical point $(x, f(x))$.

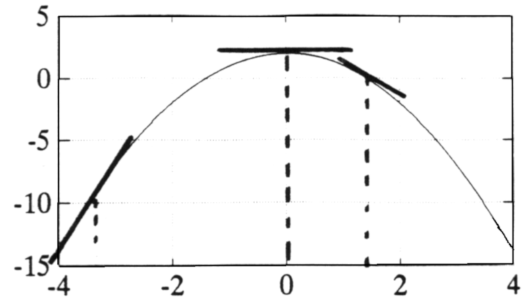
♣ 3. Consider the function $f(x) = kx$, where k is a nonzero constant. Using the limit formula, find the slope of the tangent line at a typical point $(x, f(x))$.

♣ 4. Consider the zero function $f(x) = 0$. Using the limit formula, find the slope of the tangent line at a typical point $(x, f(x))$.

EXAMPLE

using the limit formula to find the slopes of tangent lines

Next, consider the function $f(x) = -x^2 + 2$, with graph shown below.



We expect to find that:

- the slope of the tangent line at $x = 0$ is 0
- when x is small and positive, the slopes are small and negative
- when x is a large negative number, the slopes are large and positive

Let's see if this is borne out. Here, $f(x + h) = -(x + h)^2 + 2$, and we get:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{\bigl(-(x+h)^2 + 2\bigr) - (-x^2 + 2)}{h} \\
&= \lim_{h\to 0} \frac{-(x^2 + 2xh + h^2) + 2 + x^2 - 2}{h} \\
&= \lim_{h\to 0} \frac{-x^2 - 2xh - h^2 + x^2}{h} \\
&= \lim_{h\to 0} \frac{h(-2x - h)}{h} \\
&= \lim_{h\to 0} (-2x - h) \\
&= -2x
\end{aligned}$$

Observe that this is a complete mathematical sentence. For a particular value of x, the '=' signs denote equality of real numbers. Do NOT drop the limit instruction until you actually let h go to 0. This sentence shows that the limit exists for every value of x, and is equal to $-2x$. That is, the slope of the tangent line at a point $(x, f(x))$ is $-2x$.

The expected results are obtained:

- When $x = 0$, the slope of the tangent to the point $(0, 2)$ is $-2(0) = 0$, as expected.
- When $x = 0.1$, the slope of the tangent line to the point $(0.1, 1.99)$ is $-2(0.1) = -0.2$, a small negative number, as expected.
- When $x = -4$, the slope of the tangent line to the point $(-4, -14)$ is $-2(-4) = 8$, a large positive number, as expected.

EXERCISE 3	♣ 3. Graph the function $f(x) = x^2$. ♣ 2. What do you expect for the slope of the tangent line when $x = 0$? When x is a small positive number? When x is a large negative number? ♣ 3. Using the limit formula, calculate the slope of the tangent line at a typical point $(x, f(x))$. ♣ 4. What is the slope of the tangent line at $(x, f(x))$? Does this agree with your expectations?

characterizing a two-sided limit by using one-sided limits

Suppose a function g is defined both to the left and to the right of c. In order for the two-sided limit

$$\lim_{x\to c} g(x)$$

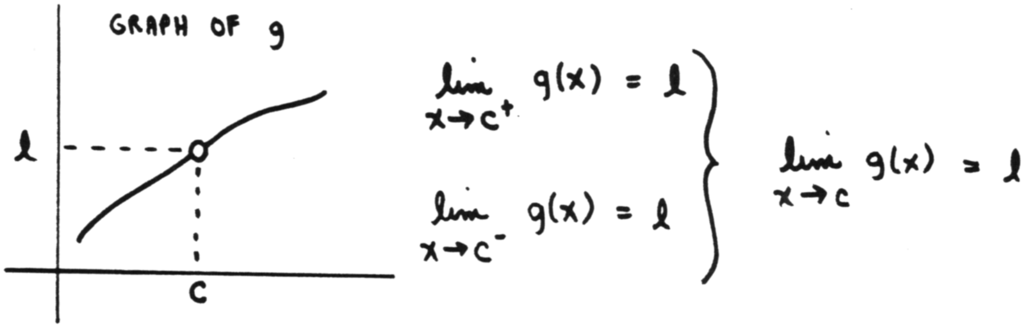
to exist, the function values $g(x)$ must approach *the same number* as x approaches c coming in from both sides.

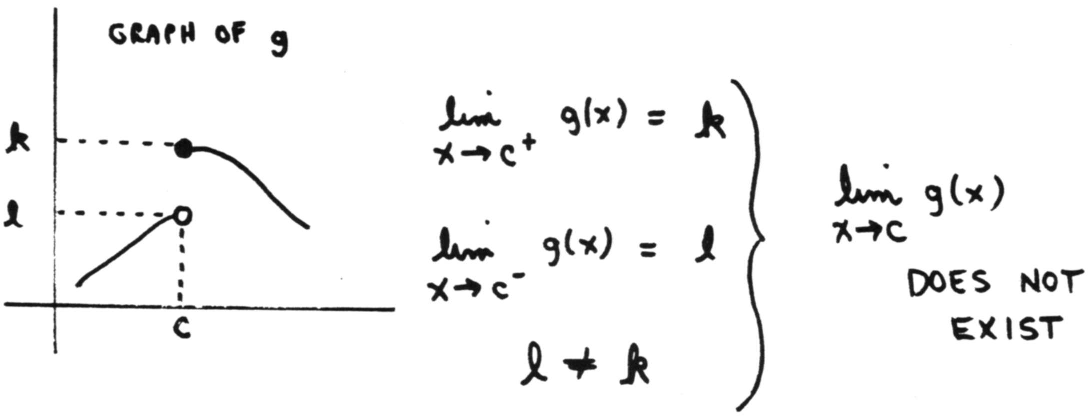
That is, the two-sided limit $\lim_{x\to c} g(x)$ exists exactly when *both* one-sided limits

$$\lim_{x\to c^+} g(x) \quad \text{and} \quad \lim_{x\to c^-} g(x)$$

exist, *and have the same value.*

This observation is used in the next examples.





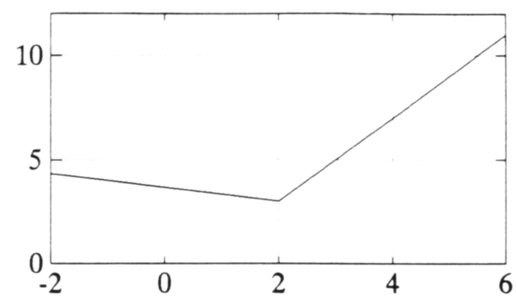
EXAMPLE

A function which does not have a tangent line at a point

Consider the function f defined piecewise as follows:

$$f(x) = \begin{cases} 2x - 1 & \text{when } x \ge 2 \\ -\frac{1}{3}x + \frac{11}{3} & \text{when } x < 2 \end{cases}$$

The graph of f is shown below.



First consider a point $(x, f(x))$ when $x > 2$. In this case, to the immediate left *and* right of the point $(x, f(x))$ the function f looks like:

$$f(x) = 2x - 1$$

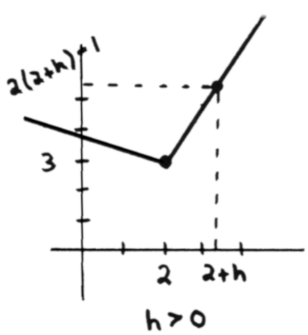
(♣ Why?) Thus, we find that:

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{\big(2(x+h) - 1\big) - (2x - 1)}{h} \\ &= \text{(♣ You fill in the details.)} \\ &= 2 \end{aligned}$$

Similarly, if $x < 2$, the slopes of tangent lines are all $-\frac{1}{3}$. (♣ Be sure to check this yourself.)

The interesting situation occurs when $x = 2$; let us now investigate the limit:

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$



Remember that this limit is, in general, a *2-sided* object. Since the function f being investigated IS defined both to the right ($h > 0$) and left ($h < 0$) of 2, we must see what happens as h approaches 0 from the right-hand side *and* the left-hand side.

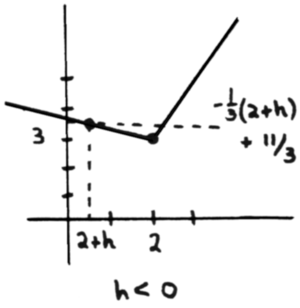
Whenever $h > 0$ (h approaches 0 from the right-hand side), we have $2 + h > 2$, so that

$$\begin{aligned} \frac{f(2+h) - f(2)}{h} &= \frac{(2(2+h) - 1) - 3}{h} \\ &= 2 \end{aligned}$$

and so:

$$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = 2$$

Whenever $h < 0$ (so that h approaches 0 from the left-hand side), we have $2 + h < 2$, so that



$$\frac{f(2+h)-f(2)}{h} = \frac{(-\frac{1}{3}(2+h)+\frac{11}{3})-3}{h}$$
$$= -\frac{1}{3}$$

and so:

$$\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} = -\frac{1}{3}$$

Since the right and left hand limits do not agree, the two-sided limit

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

does not exist.

That is, there is *no tangent line to f at $x = 2$*. This result was, of course, expected!

EXERCISE 4 Consider the function f, with graph shown below.

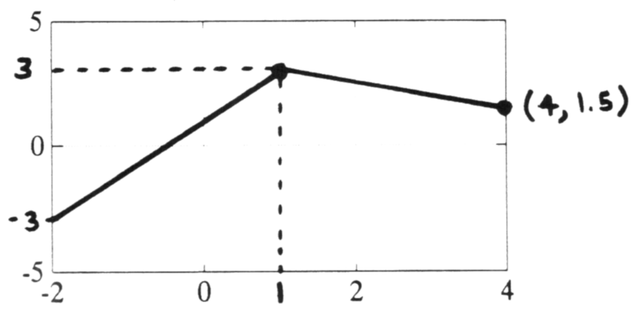
♣ 1. Give a piecewise description for this function f.

Now, attempt to find the tangent line at the point $(1, 3)$, as follows:

♣ 2. Find: $\displaystyle\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$

♣ 3. Find: $\displaystyle\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h}$

♣ 4. Does $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$ exist? Why or why not?

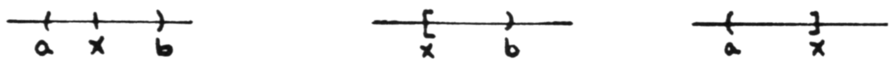


EXERCISE 5

When h is a number near zero, $x+h$ is a number near x. So, in evaluating the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} ,$$

we require that f be defined on some interval containing x. This interval can be of any of these forms:



♣ 1. If f is defined on an interval (a,b) containing x, then is the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

a genuine two-sided limit? Why or why not?

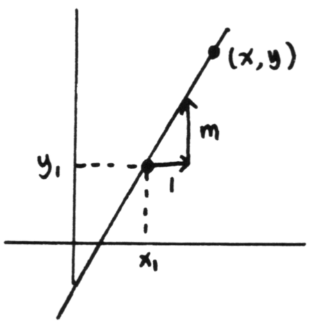
♣ 2. If f is only defined on an interval of the form $[x,b)$, then is the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ a genuine two-sided limit? If not, what type of limit is it?

♣ 3. If f is only defined on an interval of the form $(a,x]$, then is the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ a genuine two-sided limit? If not, what type of limit is it?

ALGEBRA REVIEW

point-slope form for lines

identifying lines



Two pieces of (non-contradictory, non-overlapping) information uniquely determine a line. The most common information given to identify a line is:

- two distinct (different) points on the line; or
- the slope of the line, and a point on the line.

Suppose that the slope of a line is known, call it m; and a point on the line is known, call it (x_1, y_1). Now, let (x,y) denote *any other* point on the uniquely identified line (so $x \ne x_1$). Using the points (x_1, y_1) and (x,y) to compute the (known) slope:

$$\frac{y-y_1}{x-x_1} = m \quad \iff \quad y - y_1 = m(x-x_1)$$

Thus, any point (x,y) lying on the line with slope m through (x_1,y_1) makes the equation $y-y_1 = m(x-x_1)$ true; and any point that makes the equation true lies on the line.

point-slope form of a line

That is, the equation of a line that has slope m and passes through the point (x_1, y_1) is given by:

$$y - y_1 = m(x-x_1)$$

This is called the *point-slope form* of a line.

EXAMPLE

using point-slope form

Problem: Find the equation of the line that has slope 2, and passes through the point $(-1,3)$.

Solution: The information is ideally suited to point-slope form:

$$y - 3 = 2(x-(-1)) \quad \iff \quad y = 3 + 2(x+1) \quad \iff \quad y = 2x+5$$

Problem: Find the equation of the line that passes through the points $(5, -2)$ and $(-1, 3)$.

Solution: First, find the slope of the line:

$$m = \frac{3 - (-2)}{(-1) - 5} = \frac{5}{-6} = -\frac{5}{6}$$

Then, use either point, the known slope, and point-slope form. Using the point $(5, -2)$, the equation is:

$$y - (-2) = -\frac{5}{6}(x - 5)$$

Using the point $(-1, 3)$, the equation is:

$$y - 3 = -\frac{5}{6}(x - (-1))$$

EXERCISE 6

♣ Verify that the two equations obtained above are equivalent; that is, they describe precisely the same line. That is, show that:

$$y - (-2) = -\frac{5}{6}(x - 5) \quad \iff \quad y - 3 = -\frac{5}{6}(x - (-1))$$

One way to do this is to put both equations into the same form; say, $y = mx + b$ form, or $ax + by + c = 0$ form. Once they're in the same form, they are easy to compare.

QUICK QUIZ
sample questions

1. Use a limit to compute the slope of the tangent line to the graph of $f(x) = x$ at $x = 2$. Be sure to write complete mathematical sentences.

2. In the expression $\lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$, what is the dummy variable? Rewrite the limit using a different dummy variable (you choose).

3. In the expression $\lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$, what does x represent?

4. In the limit $\lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$, what does $\frac{f(x+h)-f(x)}{h}$ represent?

5. Let $f\colon [0, 3] \to \mathbb{R}$ be defined by $f(x) = x^2$. Graph f. Does

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

exist? If so, what is it?

KEYWORDS
for this section

Tangent lines, finding the slopes of tangent lines, secant lines, difference quotient, slope of the tangent line to the graph of a function f at the point $(x, f(x))$, characterizing a two-sided limit by using one-sided limits.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

1. $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
2. $\displaystyle\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{h}$
3. $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=m$
4. $\displaystyle\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{h}=m$
5. The slope of the tangent line to the graph of $f(x)=x^2$ at the point (x,x^2) equals $2x$.
6. The slope of the tangent line to the graph of $g(x)=5$ at the point $(x,5)$ equals 0.

♣ For the remaining problems, define a function g by

$$g(h) := \frac{f(x+h)-f(x)}{h}\ ,$$

where f is a function of one variable, with $x\in\mathcal{D}(f)$.

7. Find $g(0.1)$ and $g(\Delta x)$.
8. Rewrite the limit $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ in terms of the function g.
9. When is a number h in the domain of g? Answer using a complete mathematical sentence.
10. What does the number $g(h)$ tell us?
11. What does the number $\displaystyle\lim_{h\to 0} g(h)$ tell us, when it exists?
12. Write down the ϵ-δ definition of the sentence:

$$\lim_{h\to 0} g(h) = m$$

Be sure to write a complete mathematical sentence.