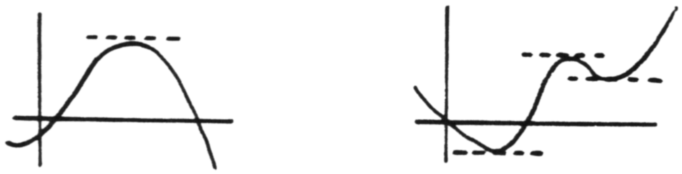
4.2 The Derivative

Introduction

In the previous section, it was shown that if a function f has a nonvertical tangent line at a point $(x, f(x))$, then its slope is given by the limit:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (*)$$

This is potentially very powerful information about the function f. For example, places where a tangent line has slope 0 often correspond to maximum or minimum values of a function.



Also, the slope of the tangent line at $(x, f(x))$ tells how the function values $f(x)$ are changing at the instant one is 'passing through' the point $(x, f(x))$: whether the graph is rising or falling, and how quickly.

Because of the importance of this slope information, the limit $(*)$, when it exists, is given a special name: it is called the *derivative of f at x*, and denoted by $f'(x)$ (read '*f prime of x*'). This is summarized below.

DEFINITIONS

differentiable at x; the derivative of f at x; differentiation

For a given function f and $x \in \mathcal{D}(f)$, if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

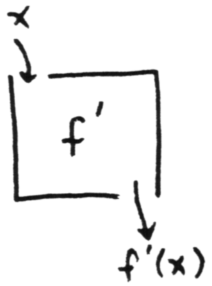
exists, then one says that f *is differentiable at x*, and writes:

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The number $f'(x)$ is called the *derivative of f at x*.

The process of finding $f'(x)$ is called *differentiation*.

DEFINITION
f',
the derivative function



Given a function f, we now have a way to construct a *new* function, named f'. This function f' is called the *derivative of* f. The domain of f' is the set of all $x \in \mathcal{D}(f)$ for which the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

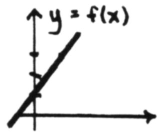
exists.

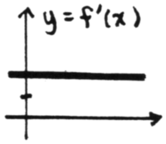
If f is differentiable at every point in its domain, then $\mathcal{D}(f') = \mathcal{D}(f)$. However, f may *not* be differentiable at every point in its domain. So, the domain of f' may be 'smaller' than the domain of f. In general, all that can be said is that $\mathcal{D}(f') \subset \mathcal{D}(f)$.

The derivative f' takes an input x, and gives as an output the slope of the tangent line to the graph of f at the point $(x, f(x))$.

EXAMPLE
differentiating
$f(x) = 2x+1$

Let f be defined by $f(x) = 2x+1$. The graph of f is a line L with slope 2. At any point on this line, the tangent line is the line L itself. So, at every point, the slope of the tangent line is 2. Thus, the function f' has the same domain as f, and is defined by $f'(x) = 2$.





One usually abbreviates the problem as follows:

PROBLEM: Differentiate: $f(x) = 2x+1$

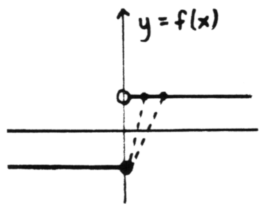
SOLUTION: $f'(x) = 2$

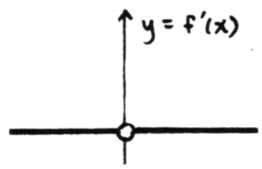
In particular, $f'(0) = 2$, $f'(\pi) = 2$, and $f'(-1002.1) = 2$.

EXAMPLE
differentiating
a function that is
defined piecewise

PROBLEM: Differentiate:

$$f(x) = \begin{cases} -1 & \text{for } x \le 0 \\ 1 & \text{for } x > 0 \end{cases}$$





SOLUTION: For all positive and negative x, tangent lines exist and have slope zero. However, f is not differentiable at 0. Intuitively, this is clear; there is no obvious way to draw a tangent line at the point $(0, -1)$. This conclusion is confirmed by investigating the right-hand limit at $x = 0$:

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{1-(-1)}{h} = \lim_{h\to 0^+} \frac{2}{h} \,,$$

which does not exist. Thus, the two-sided limit does not exist.

Summarizing:

$$f'(x) = \begin{cases} 0 & \text{for } x \ne 0 \\ \text{not defined} & \text{for } x = 0 \end{cases}$$

In particular, $f'(0.1) = 0$, $f'(-0.001) = 0$, and $f'(0)$ does not exist.

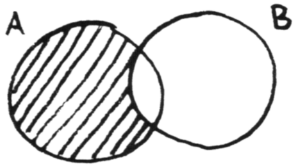
In this example, $\mathcal{D}(f) = \mathbb{R}$, and $\mathcal{D}(f') = \{x \,|\, x \ne 0\}$. Using interval notation, one can alternately write $\mathcal{D}(f') = (-\infty, 0) \cup (0, \infty)$. Unfortunately, both of these are pretty long expressions for the domain of f'. There is a simpler expression, that makes use of *set subtraction*, discussed next.

DEFINITION
set subtraction

Let A and B be sets. Define a new set, denoted by $A-B$, and read as 'A *minus* B', by:

$$A - B := \{x \mid x \in A \text{ and } x \notin B\}$$

Thus, the set $A - B$ consists of all the elements of A that are *not* elements of B. (In other words, take A and 'subtract off' any elements of B.)



EXAMPLE
set subtraction

For example, if

$$A = \mathbb{R} \quad \text{and} \quad B = [1, 2) \ ,$$

then:

$$A - B = (-\infty, 1) \cup [2, \infty) \quad \text{and} \quad B - A = \emptyset$$

If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ then:

$$A - B = \{1, 2\} \quad \text{and} \quad B - A = \{4, 5\}$$

This notation gives an easier way to describe the domain of f' in the previous example: $\mathcal{D}(f') = \mathbb{R} - \{0\}$.

EXERCISE 1

♣ 1. Why is it incorrect to say $\mathcal{D}(f') = \mathbb{R} - 0$?

For each of the following sets A and B, find both $A-B$ and $B-A$. Be sure to answer using complete mathematical sentences.

♣ 2. $A = \mathbb{R}$, $B = (-\infty, 2]$

♣ 3. $A = (-3, 3]$, $B = [-1, 4)$

♣ 4. $A = \mathbb{R}$, B is the set of irrational numbers

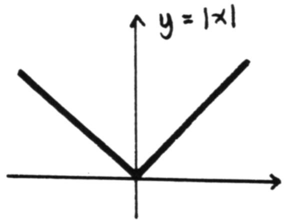
Find sets A and B so that $S = A - B$. (There is *not* a unique correct answer.)

♣ 5. $S = (-1, 0) \cup (0, 1]$

♣ 6. $S = \{1, 2, 3\}$

EXAMPLE

differentiating $f(x) = |x|$; a function that is continuous at a point, but not differentiable there



$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Consider the function $f(x) = |x|$.

For $x > 0$, $f(x) = |x| = x$ is certainly differentiable with derivative $f'(x) = 1$.

For $x < 0$, $f(x) = |x| = -x$ is also differentiable with derivative $f'(x) = -1$.

When $x = 0$, there is no tangent line. In this case, both one-sided limits exist, but do not agree:

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0^+} \frac{|h| - |0|}{h} \\ &= \lim_{h \to 0^+} \frac{h}{h} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \lim_{h \to 0^-} \frac{|0+h| - |0|}{h} &= \lim_{h \to 0^-} \frac{|h|}{h} \\ &= \lim_{h \to 0^-} \frac{-h}{h} \\ &= -1 \end{aligned}$$

Since the one-sided limits do not agree, the two-sided limit *does not exist*, and f is *not* differentiable at $x = 0$.

Summarizing:

$$f'(x) = \begin{cases} 1 & \text{for } \ x > 0 \\ \text{not defined} & \text{for } \ x = 0 \\ -1 & \text{for } \ x < 0 \end{cases}$$

Thus, for example, $f'(1.7) = 1$, $f'(-4/3) = -1$, and $f'(0)$ does not exist.

EXERCISE 2

Let $f(x) = 3x - 1$.

♣ 1. Graph f. What is $\mathcal{D}(f)$?

♣ 2. What is the function f'? In particular, what is $\mathcal{D}(f')$?

Now, let $f(x) = |x - 3|$.

♣ 3. Give a piecewise description for f.

♣ 4. Graph f. What is $\mathcal{D}(f)$?

♣ 5. For $x > 3$, what is $f'(x)$?

♣ 6. For $x < 3$, what is $f'(x)$?

♣ 7. Show that f' is not defined at $x = 3$, by investigating the limits:

$$\lim_{h \to 0^+} \frac{f(3+h) - f(3)}{h} \quad \text{and} \quad \lim_{h \to 0^-} \frac{f(3+h) - f(3)}{h}$$

♣ 8. Write down a piecewise description of f'.

♣ 9. Graph f'.

EXAMPLE

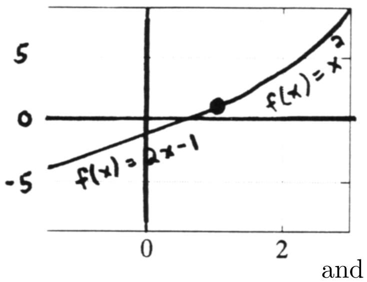
a 'patched together' function that IS differentiable at the patching point

PROBLEM: Is the function

$$f(x) = \begin{cases} x^2 & x \ge 1 \\ 2x - 1 & x < 1 \end{cases}$$

differentiable at $x = 1$?

SOLUTION: Note that $f(1) = 1^2 = 1$. Investigate both one-sided limits:



$$\begin{aligned} \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0^+} \frac{(1+h)^2 - 1}{h} \\ &= \lim_{h \to 0^+} \frac{1 + 2h + h^2 - 1}{h} \\ &= \lim_{h \to 0^+} \frac{h(2+h)}{h} = 2 \end{aligned}$$

and

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{(2(1+h) - 1) - 1}{h} = \lim_{h \to 0^-} \frac{2h}{h} = 2$$

Since both one-sided limits agree,

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

exists and equals 2. Thus, f is differentiable at 1, and $f'(1) = 2$. That is, the tangent line to the graph of f at the point $(1, 1)$ has slope 2.

EXERCISE 3

Let f be defined by:

$$f(x) = \begin{cases} x^2 & \text{for } x \le -1 \\ -2x - 1 & \text{for } x > -1 \end{cases}$$

♣ 1. Graph f. What is $\mathcal{D}(f)$?

♣ 2. Does $f'(x)$ exist for $x < -1$? If so, what is it?

♣ 3. Does the limit $\lim_{x \to -1^-} f'(x)$ exist? If so, what is it?

♣ 4. Does $f'(x)$ exist for $x > -1$? If so, what is it?

♣ 5. Investigate two appropriate one-sided limits to decide if $f'(-1)$ exists. If it does, what is it?

♣ 6. Is there a tangent line to the graph of f at the point with x-value -1? If so, what is its slope?

♣ 7. Graph f'. What is $\mathcal{D}(f')$?

EXAMPLE

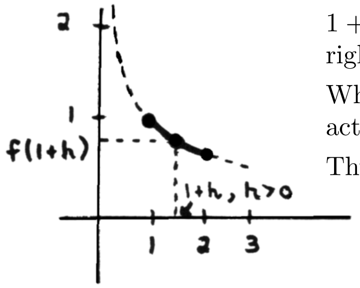
a function that is differentiable at an endpoint of its domain

PROBLEM: Consider the function $f\colon [1,2] \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Is f differentiable at $x = 1$?

SOLUTION: Note that $f(1) = \frac{1}{1} = 1$. For f to be differentiable at $x = 1$, the limit

$$\lim_{h\to 0} \frac{f(1+h)-1}{h}$$

must exist. Does it? Remember that to investigate this limit, one only considers values of h that are *close to* 0 and *in the domain of the function* $\frac{f(1+h)-1}{h}$. When is h in the domain of $\frac{f(1+h)-1}{h}$? Only when $1+h \in \mathcal{D}(f)$. And for this function, $1+h \in \mathcal{D}(f)$ only when $h > 0$. Here, the 'two-sided limit' is identical to the right-hand limit.



$f\colon [1,2] \to \mathbb{R}$,
$f(x) = \frac{1}{x}$

When a function is only defined on one side of a point, the 'two-sided limit' is actually just a one-sided limit.

Thus, one has:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0^+} \frac{\frac{1}{1+h} - 1}{h} && \text{(line 1)}\\
&= \lim_{h\to 0^+} \frac{\frac{1}{1+h} - \frac{1+h}{1+h}}{h} && \text{(line 2)}\\
&= \lim_{h\to 0^+} \frac{1-1-h}{h(1+h)} && \text{(line 3)}\\
&= \lim_{h\to 0^+} \frac{-1}{1+h} = -1 && \text{(line 4)}
\end{aligned}$$

Thus, f is differentiable at $x = 1$, and $f'(1) = -1$.

EXERCISE 4

♣ 1. Give a reason (or reasons) for each line of the preceding display. The lines are numbered for easy reference.

Consider the function $f\colon [0,4] \to \mathbb{R}$ given by $f(x) = (x-1)^2$.

♣ 2. Graph f.

♣ 3. Is f differentiable at $x = 0$? That is, does the limit

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

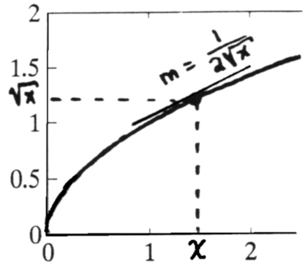
exist? Justify your answer. Be sure to write complete mathematical sentences.

EXAMPLE

a function that is not differentiable at an endpoint of its domain

Let $f(x) = \sqrt{x}$.

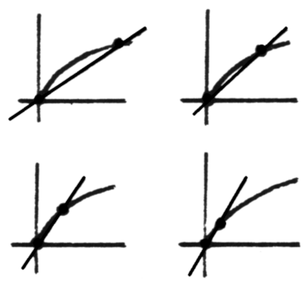
For all $x > 0$ one has:



$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} && \text{(line 1)} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} && \text{(line 2)} \\
&= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} && \text{(line 3)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} && \text{(line 4)} \\
&= \frac{1}{2\sqrt{x}} && \text{(line 5)}
\end{aligned}$$

Thus, at each point $(x, \sqrt{x})$ for $x > 0$, the tangent line exists and has slope $\frac{1}{2\sqrt{x}}$.

EXERCISE 5	♣ Give a reason (or reasons) for each line in the display above. The lines are numbered for easy reference.



Now think about what happens when $x = 0$. In this case, the limit is actually a right-hand limit, and reduces to:

$$\lim_{h\to 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h\to 0^+} \frac{\sqrt{h}}{h}$$

For $h > 0$, we can write $h = (\sqrt{h})^2$, so that:

$$\frac{\sqrt{h}}{h} = \frac{\sqrt{h}}{(\sqrt{h})^2} = \frac{1}{\sqrt{h}}$$

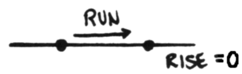
But as $h \to 0^+$, $\frac{1}{\sqrt{h}}$ does *not* approach a specific real number. It gets arbitrarily large. There is a vertical tangent line at the point $(0, 0)$, and a vertical line has no slope. So f is *not* differentiable at 0.

Caution!

'no slope' versus 'zero slope'

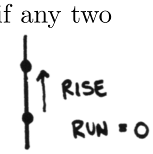
Every horizontal line has zero slope. Choosing any two points on the line, and traveling from one point to the other via the rule '*rise, then run*' yields:

$$\frac{\text{rise}}{\text{run}} = \frac{0}{\text{some nonzero number}} = 0$$



Every vertical line has no slope; that is, the slope is undefined. For if any two points are chosen on the line, computation of the slope yields

$$\frac{\text{rise}}{\text{run}} = \frac{\text{some nonzero number}}{0} ,$$

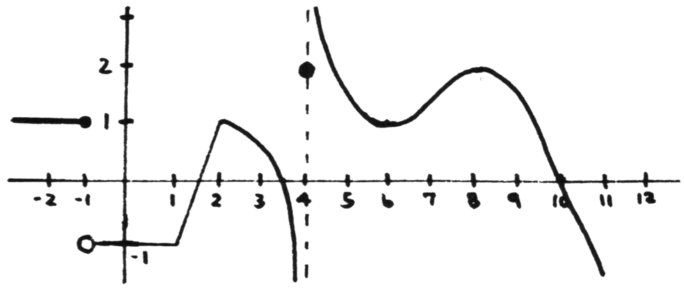


and division by zero is undefined.

Thus, *no slope* and *zero slope* have entirely different meanings. This can be confusing, because in English, the words 'no' and 'zero' are often used as synonyms.

reading info from a graph

Consider the function f whose graph is shown below:



Read the following information from the graph, if possible. If a quantity does not exist, so state.

$$f(-1), \quad f'(-1), \quad f'(-1.5), \quad f'(1.5), \quad f'(2), \quad f(4), \quad f'(6), \quad f'(7)$$

SOLUTION:

- $f(-1) = 1$
- $f'(-1)$ does not exist
- $f'(-1.5) = 0$
- $f'(1.5) = 2$ (Use the known points $(1, -1)$ and $(2, 1)$ to compute the slope.)
- $f'(2)$ does not exist
- $f(4) = 2$
- $f'(6) = 0$
- $f'(7) > 0$; one might estimate that $f'(7) \approx 1$

Now, answer the following questions about f:

- What is $\mathcal{D}(f)$?
- What is $\mathcal{R}(f)$?
- Where is f continuous?
- Where is f differentiable?
- What is $\{x \mid f(x) > 0\}$?
- What is $\{x \mid f(x) \in [-1, 1]\}$?
- What is $\{x \mid f(x) = 1\}$?

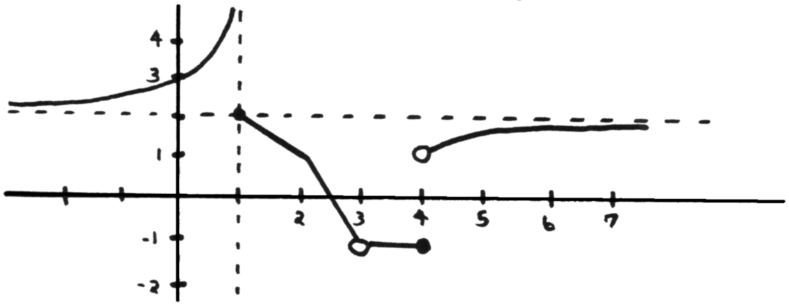
SOLUTION:

For all these answers, the assumption is made that the patterns indicated at the four borders of the graph continue.

- $\mathcal{D}(f) = \mathbb{R}$
- $\mathcal{R}(f) = \mathbb{R}$
- f is continuous at all x in the set $(-\infty, -1) \cup (-1, 4) \cup (4, \infty)$. A simpler notation for this set is $\mathbb{R} - \{-1\,,\,4\}$.
- f is differentiable at all x in the set $\mathbb{R} - \{-1\,,\,1\,,\,2\,,\,4\}$.
- Some approximation is necessary here.
  $\{x \,|\, f(x) > 0\} = (-\infty, -1] \cup (1.5, 3.5) \cup [4, 10)$
- Some approximation is necessary here.
  $\{x \,|\, f(x) \in [-1, 1]\} = (-\infty, 3.8) \cup \{6\} \cup (9.5, 10.5)$
- Some approximation is necessary here.
  $\{x \,|\, f(x) = 1\} = (-\infty, -1] \cup \{2, 6, 9.5\}$

EXERCISE 6

Consider the function f whose graph is shown below. Read the following information from the graph, if possible. Approximate, when necessary. If a quantity does not exist, so state. Be sure to write complete mathematical sentences.



♣ 1. $f(0)$, $f(1)$, $f'(1)$, $f'(2)$, $f'(1.34)$,
$f(3)$, $f(4)$, $f'(\pi)$, $f'(1000)$

♣ 2. What is $\mathcal{D}(f)$?

♣ 3. What is $\mathcal{D}(f')$?

♣ 4. What is $\mathcal{R}(f)$?

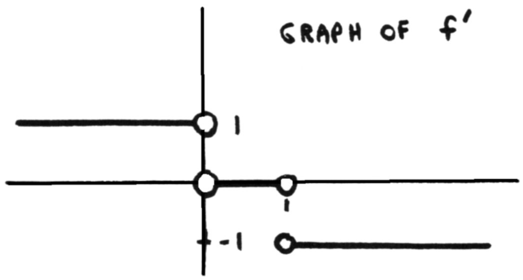
♣ 5. Where is f continuous? Classify any discontinuities.

♣ 6. What is $\{x \,|\, f(x) \le 0\}$?

♣ 7. What is $\{x \,|\, f'(x) < 0\}$?

'reconstructing' a function from its derivative

Suppose that a function f has *derivative* f' whose graph is shown below. What, if anything, can be said about the graph of f?



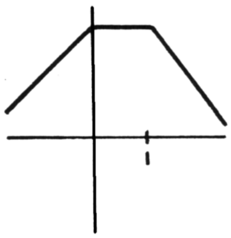
SOLUTION:

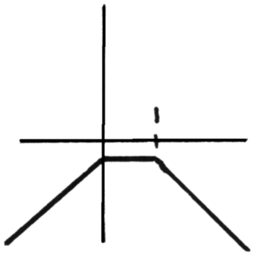
For $x < 0$, the tangent lines to the graph of f must all have slope 1.

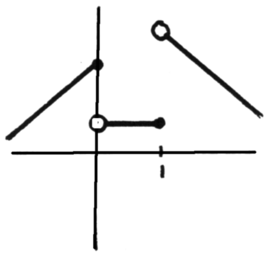
For $0 < x < 1$, the tangent lines to the graph of f must all have slope 0.

For $x > 1$, the tangent lines to the graph of f must all have slope -1.

There *is not* a unique function f that satisfies these requirements. For example, any of the following graphs would produce the specified derivative:

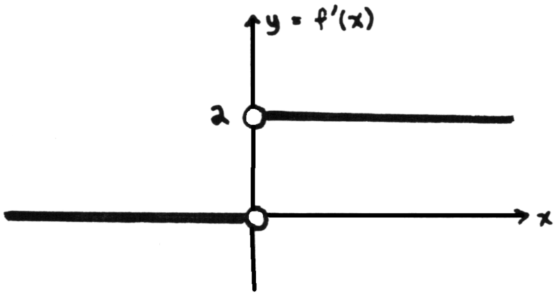






EXERCISE 7

Suppose that a function f has derivative f' whose graph is shown below:

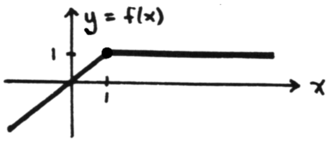


♣ 1. What, if anything, can be said about the graph of f?

♣ 2. Graph three different functions f that could have the specified derivative.

QUICK QUIZ
sample questions

1. Give a precise definition of $f'(x)$.
2. What is the difference between f' and $f'(x)$?
3. If $A = [0, 4)$ and $B = \{0, 2, 4\}$, find $A - B$ and $B - A$.
4. For the function given below, find and graph f'.



5 TRUE or FALSE: If f is differentiable at x, then f is defined at x.

KEYWORDS
for this section

Differentiable at x, $f'(x)$ is the derivative of f at x, differentiation, f' is the derivative function, finding derivatives using the definition, set subtraction.

END-OF-SECTION EXERCISES

For each function f listed below, do the following:

♣ Graph f. What is $\mathcal{D}(f)$?

♣ Find f'. When necessary, use the definition of derivative.

♣ Graph f'. What is $\mathcal{D}(f')$?

1. $f(x) = |x - 2|$
2. $f(x) = \begin{cases} 2 & \text{for } \ -3 < x \le 0 \\ \frac{1}{x} & \text{for } \ 0 < x < 4 \end{cases}$
3. $f(x) = \begin{cases} x^2 & \text{for } \ x \le 1 \\ 2x & \text{for } \ x > 1 \end{cases}$

♣ Use the definition of the derivative to find $f'(c)$ for each function f and number $c \in \mathcal{D}(f)$.

4. $f(x) = 3x^2 - 1, \ \ c = 2$
5. $f(x) = \frac{1}{x-1}, \ \ c = 2$
6. $f(x) = \sqrt{x} + 1, \ \ c = 4$

♣ Find the equation of the tangent line to the graph of the function f at the specified point. Feel free to use any earlier results.
The point-slope form may be useful: remember that

$$y - y_1 = m(x - x_1)$$

is the equation of the line that has slope m and passes through the point (x_1, y_1).

7. $f(x) = x^2, \ \ c = 3$
8. $f(x) = \sqrt{x}, \ \ c = 0$
9. $f(x) = (x + 2)^2 + 1, \ \ c = -2$