4.8 Implicit Differentiation
(Optional)

Introduction;

$y = f(x)$
explicit representation

You are used to seeing equations of the form:

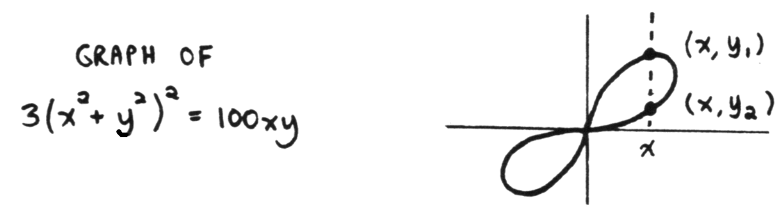
$$y = f(x)$$

Here, y is isolated on one side of the equation, and all the x's appear on the other side. In such a case, one says that y is given *explicitly* in terms of x. When such a representation is possible, y is truly a *function* of x; once a choice for x is made, substitution into the formula $f(x)$ yields the corresponding unique value of y.

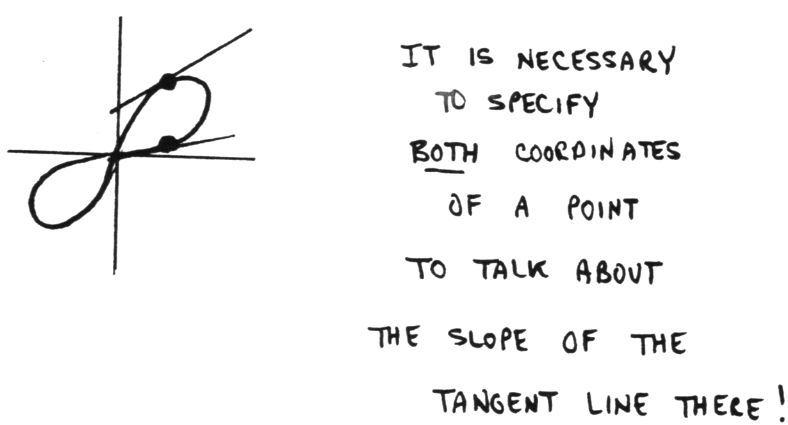
implicit representation

Often, it is *inconvenient* or *impossible* to solve for y in terms of x. In many such instances, the inability to solve uniquely for y in terms of x stems from the fact that y is *not* a function of x.

For example, the graph of $3(x^2 + y^2)^2 = 100xy$ is shown below. Although y is *not* a function of x, one can still talk about the slopes of tangent lines at various points on the graph. However, *since we are not dealing with a function*, to specify the location in the graph in which there is interest, it is necessary to specify *both* an x and y value.



If a relationship between x and y is such that y is not solved explicitly in terms of x, then one says that y is expressed *implicitly* in terms of x.

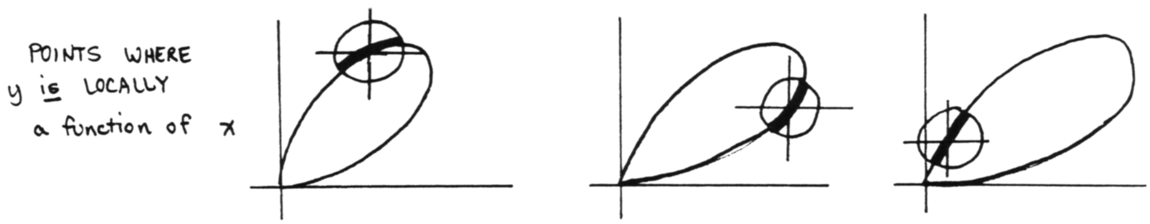


y is locally a function of x

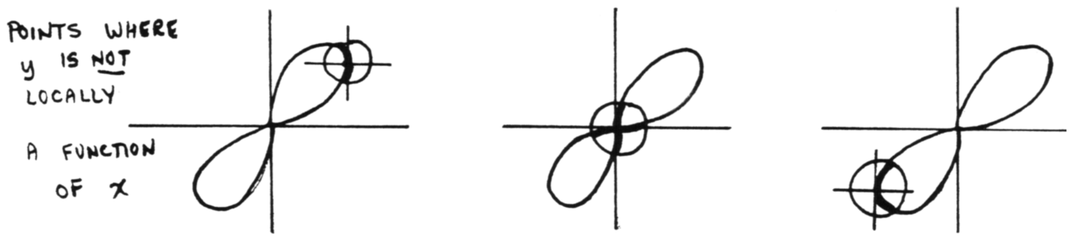
The technique of *implicit differentiation* is used to get information about slopes of tangent lines, in cases when y is given *implicitly* in terms of x. The key idea is this: although y is not (globally) a function of x, if attention is restricted to a *local* situation, then y *CAN* be viewed as a function of x (at most points).

Think about it this way: take a 'mini' coordinate system, and center the origin at a point on a curve. If it is possible to draw a circle (no matter how small!) around this coordinate system, within which one sees the graph of a *function*, then, locally, y is a function of x.

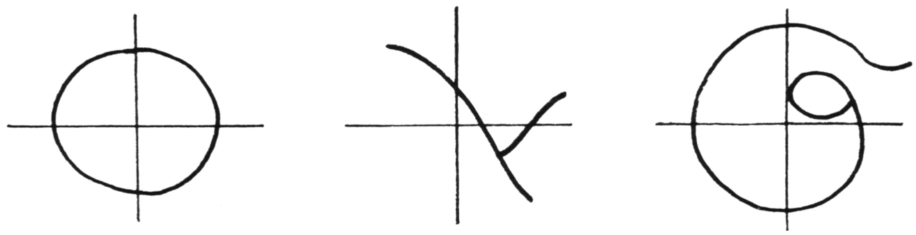
The sketches below show several points at which y IS locally a function of x.



The sketches below show three points at which y is NOT locally a function of x. No matter how small a circle is drawn around the point, there is no way to enclose a piece of graph for which y is a function of x.



EXERCISE 1 ♣ On the graphs below, identify any points where y is NOT locally a function of x.



the technique of implicit differentiation

Implicit differentiation works like this: given a relationship between x and y, differentiate both sides of the equation with respect to x, remembering that (locally, at least!) y *is* a function of x.

if y is a function of x, then it must be differentiated accordingly

Suppose that y is a function of x, say $y = y(x)$. Then, *y must be differentiated using the rules that are appropriate for differentiating functions of x.* For example:

$$\frac{d}{dx}y^3 = \frac{d}{dx}(y(x))^3 = 3(y(x))^2 \cdot y'(x)$$

This is usually written more simply as:

$$\frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}$$

Similarly:

$$\frac{d}{dx}x \ln y = x(\frac{1}{y})\frac{dy}{dx} + \ln y$$

EXERCISE 2

Find the following derivatives, treating y as a function of x.

♣ 1. $\frac{d}{dx}(y^2)$

♣ 2. $\frac{d}{dx}(xy)$

♣ 3. $\frac{d}{dx}(x+y)^3$

♣ 4. $\frac{d}{dx}(\ln y)$

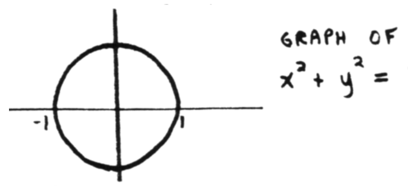
when y is a function of x, the formula for $\frac{dy}{dx}$ is also a function of x

Whenever y is a (global) function of x, then each point on the curve is uniquely identified by its x-coordinate. In particular, if one wants to talk about the slope of a tangent line at a point, it is only necessary to specify the x-coordinate to locate the point. Therefore, whenever y is a function of x, $\frac{dy}{dx}$ is also a function of x.

However, if y is NOT a function of x, then to identify a point on the curve, BOTH its x and y coordinates are needed. So, to talk about the slope of a tangent line at a particular point, one also needs to specify both coordinates. In such cases, then, the formula for $\frac{dy}{dx}$ involves BOTH x AND y.

EXAMPLE

Consider the equation $x^2 + y^2 = 1$. The set of all points (x, y) that make this equation true is the circle of radius 1, centered at the origin. (See the Algebra Review on circles at the end of this section.)



Observe that y is not (globally) a function of x. However, *at all points except* $(1, 0)$ *and* $(-1, 0)$, y is *locally* a function of x.

Differentiating both sides of $x^2 + y^2 = 1$ with respect to x, and remembering that (at least locally) y is a function of x, yields:

$$2x + 2y\frac{dy}{dx} = 0$$

In this case, it is possible to solve for $\frac{dy}{dx}$:

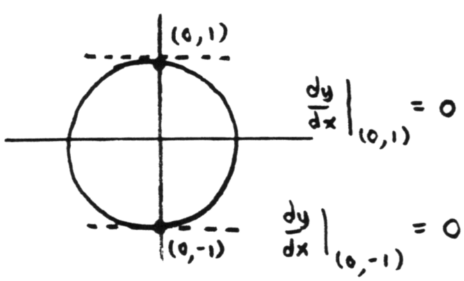
$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

Observe that this formula for $\frac{dy}{dx}$ depends on *both* x and y. This was expected, since *both* an x and y coordinate are needed to uniquely identify the point where the slope of the tangent line is desired.

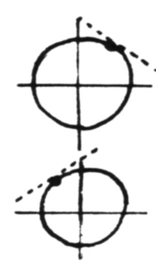
The formula seems to yield reasonable results. For example, $\frac{dy}{dx}|_{(0,1)} = -\frac{0}{1} = 0$. This information reflects the fact that the slope of the tangent line at the point $(0, 1)$ is horizontal.

Also, $\frac{dy}{dx}|_{(0,-1)} = -\frac{0}{-1} = 0$. Again, the tangent line at $(0, -1)$ is horizontal.

Some additional examples are given below. Note in particular that the formula for the derivative *fails* when $y = 0$; there are *vertical* tangent lines at these points.



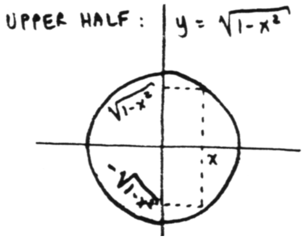
x	y	$\frac{dy}{dx}$
$1/2$	$\sqrt{3}/2$	$-1/\sqrt{3}$
$-1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$



same example, different viewpoint

In the previous example, the equation $x^2 + y^2 = 1$ *could have* been solved for y, to obtain:

$$y = \pm\sqrt{1 - x^2}$$



Here, the '+' sign yields the upper half of the circle, and the '$-$' sign the lower half of the circle. Differentiating $y = +\sqrt{1 - x^2}$ in the normal way yields the slopes of the tangent lines to the upper half of the circle:

$$\frac{dy}{dx} = \frac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x) = -\frac{x}{\sqrt{1 - x^2}} = -\frac{x}{y}$$

Thus, the formula is compatible with that obtained by implicit differentiation. However, differentiating implicitly was much easier than this latter approach.

EXERCISE 3

♣ Differentiate $y = -\sqrt{1 - x^2}$ to get a formula for $\frac{dy}{dx}$ that is valid for the lower half of the circle. Show that the result is compatible with the formula obtained by differentiating implicitly.

EXERCISE 4

♣ 1. Graph the equation $(y - 2)^2 + x^2 = 9$.

♣ 2. At what points on the graph is y NOT locally a function of x?

♣ 3. Find $\frac{dy}{dx}$ by differentiating implicitly. At what point(s) does the formula fail? Why?

further uses for implicit differentiation

There are two other common situations where implicit differentiation is extremely useful. These are discussed next.

differentiating complicated products & quotients

Recall that the log of a product is the sum of the logs; the log of a quotient is the difference of the logs. Since differentiating sums and differences is *much easier* than differentiating products and quotients, we can exploit the logarithm as illustrated in the next example.

EXAMPLE

logarithmic differentiation

Problem: Differentiate $y = \dfrac{x^2(x-2)}{\sqrt{2x-3}}$.

Solution: First, find the natural logarithm of y :

$$\begin{aligned}\ln y &= \ln(x^2(x-2)) - \ln\sqrt{2x-3} \\ &= \ln x^2 + \ln(x-2) - \ln(2x-3)^{1/2} \\ &= 2\ln x + \ln(x-2) - \frac{1}{2}\ln(2x-3)\end{aligned}$$

In the equation

$$\ln y = 2\ln x + \ln(x-2) - \frac{1}{2}\ln(2x-3)\ ,$$

y is given *implicitly* as a function of x. Implicit differentiation yields:

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{x-2} - \frac{1}{2}\cdot\frac{1}{2x-3}\cdot 2$$

Since y truly is a function of x in this example, we expect to be able to get a formula for the derivative as a function of x, and we certainly can:

$$\begin{aligned}\frac{dy}{dx} &= y\cdot\left[\frac{2}{x} + \frac{1}{x-2} - \frac{1}{2x-3}\right] \\ &= \frac{x^2(x-2)}{\sqrt{2x-3}}\left[\frac{2}{x} + \frac{1}{x-2} - \frac{1}{2x-3}\right]\end{aligned}$$

This process of differentiating a function y by first taking the logarithm and then using implicit differentiation is often referred to as *logarithmic differentiation.*

EXERCISE 5

Use logarithmic differentiation to differentiate:

♣ 1. $y = (\frac{1}{x})(\frac{1}{2x-1})(\frac{1}{3x-1})$

♣ 2. $y = \dfrac{x^4\sqrt[3]{x-1}}{\sqrt[5]{2x+1}}$

differentiating variable expressions to variable powers; logarithmic differentiation

Another common use for implicit differentiation is in differentiating variable expressions raised to variable powers, illustrated next.

Suppose that $y = x^{2x}$. The extended power rule for differentiation *does not apply* here, since the exponent is *not* a constant. Instead, find the natural logarithm of y ,

$$\ln y = \ln x^{2x} = 2x\ln x$$

and then differentiate implicitly:

$$\frac{1}{y}\frac{dy}{dx} = 2x\frac{1}{x} + (2)(\ln x) = 2(1+\ln x)$$

Since y *is* truly a function of x, we expect to be able to express the derivative as a function of x, and we can:

$$\frac{dy}{dx} = y\cdot 2(1+\ln x) = 2x^{2x}(1+\ln x)$$

EXERCISE 6

Use logarithmic differentiation to differentiate. In each case, write $\frac{dy}{dx}$ as a function of x.

♣ $y = x^x$

♣ $y = (2x)^x$

♣ $y = (2x)^{3x}$

♣ $y = (\sqrt{x+1})^{(x^2)}$

ALGEBRA REVIEW

circles

EXERCISE 7

the relationship between the sentences $a = b$ and $a^2 = b^2$

Consider the equations $a = b$ and $a^2 = b^2$.

♣ 1. Show that these equations are NOT equivalent. That is, find choices for a and b for which the sentences $a = b$ and $a^2 = b^2$ have different truth values.

♣ 2. Now consider the sentence:

$$\text{For } a \ge 0 \text{ and } b \ge 0, \quad a = b \iff a^2 = b^2 \,.$$

The phrase 'For ...' has been used to restrict the universal sets for a and b to the nonnegative real numbers. This sentence asserts that, as long as both a and b are nonnegative, then the equations $a = b$ and $a^2 = b^2$ WILL always have the same truth values. Convince yourself that this is true.

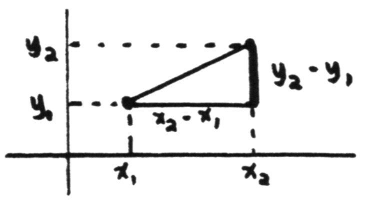
♣ 3. Conclude the following: if you are in a situation where it is known that both a and b are nonnegative, then the sentence $a = b$ can be replaced, if convenient, by the equation $a^2 = b^2$.

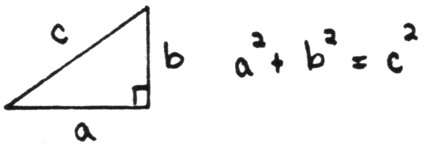
distance between two points

Recall first that the distance between points (x_1, y_1) and (x_2, y_2) is given by:

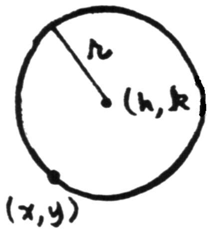
$$\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

This formula is an immediate consequence of Pythagorean's Theorem.





circles



Now, it is desired to find the equation of the circle with center (h, k) and radius r. That is, we seek an equation that is true for all points (x, y) that lie on the circle of radius r centered at the point (h, k).

This is easy to get: we want those points (x, y) whose distance from (h, k) is equal to r. That is, we want points (x, y) satisfying:

$$\sqrt{(y-k)^2 + (x-h)^2} = r$$

Since both sides of this equation are nonnegative (r is the radius of a circle, and square roots are nonnegative), an equivalent equation is obtained by squaring both sides (see Exercise #7):

$$(y-k)^2 + (x-h)^2 = r^2$$

This is the equation of the circle centered at (h, k), with radius r.

EXAMPLE

Problem: Graph $x^2 + y^2 = 1$.

Solution: Rewrite:

$$x^2 + y^2 = 1 \quad \iff \quad (x-0)^2 + (y-0)^2 = 1^2$$

This is the circle centered at $(0, 0)$ with radius 1 .

Problem: Graph $(3-y)^2 + (x+1)^2 = 4$.

Solution: Rewrite:

$$(3-y)^2 + (x+1)^2 = 4 \quad \iff \quad (y-3)^2 + (x-(-1))^2 = 2^2$$

This is the circle centered at $(-1, 3)$ with radius 2 .

Problem: Graph $x^2 + y^2 + 3y = \frac{7}{4}$.

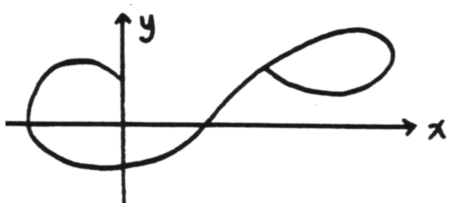
Solution: Rewrite, by completing the square:

$$\begin{aligned} x^2 + y^2 + 3y = \frac{7}{4} \quad &\iff \quad x^2 + (y^2 + 3y + (\frac{3}{2})^2) = \frac{7}{4} + (\frac{3}{2})^2 \\ &\iff \quad x^2 + (y + \frac{3}{2})^2 = \frac{7}{4} + \frac{9}{4} \\ &\iff \quad x^2 + (y - (-\frac{3}{2}))^2 = 2^2 \end{aligned}$$

This is the circle centered at $(0, -\frac{3}{2})$ with radius 2 .

QUICK QUIZ
sample questions

1. Let $xy^2 = 2$. Find $\frac{dy}{dx}$, by differentiating implicitly.
2. Let $y = x^{2x}$. Find y', by using logarithmic differentiation.
3. Graph $x^2 - 2x + y^2 = 8$.
4. Write an equation where y is given *explicitly* in terms of x; where y is given *implicitly* in terms of x.
5. On the sketch below, identify any point(s) where y is NOT locally a function of x.



KEYWORDS
for this section

Explicit versus implicit representations, y is locally a function of x, implicit differentiation, logarithmic differentiation, differentiating complicated products and quotients, differentiating variable expressions to variable powers, equations of circles.

END-OF-SECTION EXERCISES

♣ Graph the equation (each is a circle).

♣ Identify any point where y is NOT locally a function of x.

♣ Find y' by differentiating implicitly.

♣ Check that the given point(s) lie on the circle; write the equation of the tangent line at these points.

1. $x^2 + 4x + y^2 - 2y + 4 = 0$; $(-2, 2)$, $(-1, 1)$
2. $x^2 + 4x + y^2 - 2y = -4$; $(-2, 0)$, $(-3, 1)$
3. $4x - 2y = -x^2 - y^2 - 1$; $(-1, 1 + \sqrt{3})$
4. $4x - 2y = -x^2 - y^2 - 1$; $(-1, 1 - \sqrt{3})$