5.1 Increasing and Decreasing Functions

increasing and decreasing functions; roughly

Roughly, a function f is *increasing* if its graph moves UP, traveling from left to right; and is *decreasing* if its graph moves DOWN, traveling from left to right. The precise definitions follow.

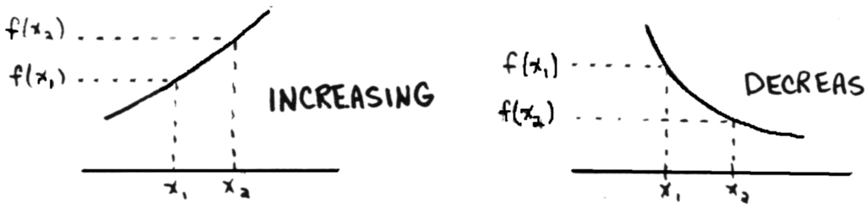
DEFINITION
increasing and decreasing functions

A function f is *increasing* on an interval I if and only if:

$$\text{for all } x_1, x_2 \in I, \quad x_1 < x_2 \implies f(x_1) < f(x_2)$$

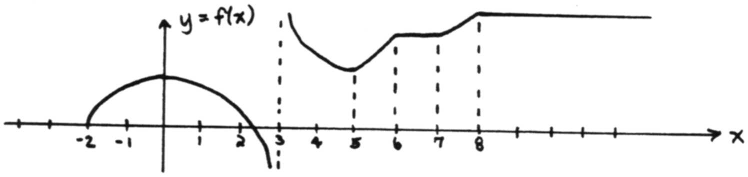
A function f is *decreasing* on an interval I if and only if:

$$\text{for all } x_1, x_2 \in I, \quad x_1 < x_2 \implies f(x_1) > f(x_2)$$



EXAMPLE

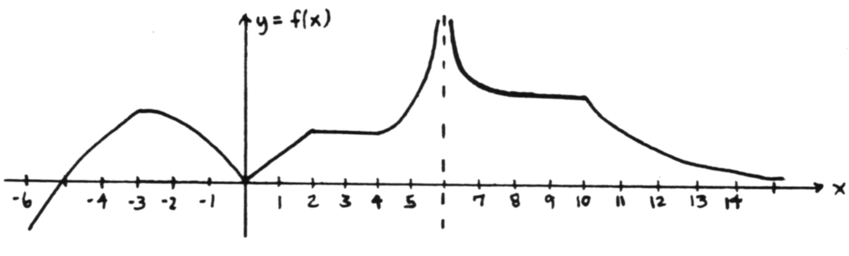
Problem: Identify the open intervals on which the function graphed below is increasing and decreasing.



Solution:
f increases on $(-2, 0) \cup (5, 6) \cup (7, 8)$
f decreases on $(0, 3) \cup (3, 5)$
f is neither increasing nor decreasing on $(6, 7) \cup (8, \infty)$

EXERCISE 1

♣ Identify the open intervals on which the function graphed below is increasing and decreasing. Be sure to write complete mathematical sentences.



EXERCISE 2

Sketch the graphs of functions satisfying the following properties:

♣ 1. f increases on $(1,3)$, and $f(x) < 0 \ \ \forall\, x \in (1,3)$

♣ 2. f increases on $(1,3)$, $f(2) = 0$, and $f(x) < -5 \ \ \forall\, x \in (1,1.5)$

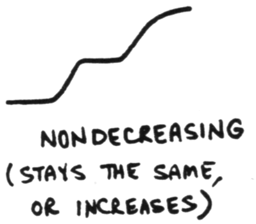
♣ 3. f increases on $(1,3)$, and is not differentiable at $x = 2$

♣ 4. f increases on $(1,3)$, decreases on $(3,5)$, and is differentiable at $x = 3$ What do you suspect that the number $f'(3)$ must be?

♣ 5. f increases on $(1,3)$, decreases on $(3,5)$, and is not differentiable at $x = 3$

EXERCISE 3

nonincreasing, nondecreasing functions



Sometimes it is important to know where a function *doesn't decrease.* A function *doesn't decrease* if it either increases or stays the same, and functions satisfying this property are called *nondecreasing* functions. Here's a precise definition:

A function f is *nondecreasing* on an interval I if and only if:

$$\text{for all } x_1, x_2 \in I, \quad x_1 < x_2 \implies f(x_1) \le f(x_2)$$

When you see the word 'nondecrease', think to yourself: 'does not decrease'.

♣ 1. How should you read aloud '$x_1 < x_2 \implies f(x_1) \le f(x_2)$'? What is the hypothesis of this implication? What is the conclusion?

♣ 2. Let $x_1 = 1$, $x_2 = 3$, $f(x_1) = -1$ and $f(x_2) = -0.5$. For these choices:

a Is the hypothesis of $x_1 < x_2 \implies f(x_1) \le f(x_2)$ true or false?

b Is the conclusion of $x_1 < x_2 \implies f(x_1) \le f(x_2)$ true or false?

c Is the sentence $x_1 < x_2 \implies f(x_1) \le f(x_2)$ true or false?

♣ 3. Now let $x_1 = 1$, $x_2 = 3$, $f(x_1) = -0.5$ and $f(x_2) = -1$. Answer the same questions as in (2).

♣ 4. Let $I = (0,4)$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $f(1) = 1$, $f(2) = 2$, $f(3) = 1$. Based on this information alone, can the truth of the sentence

$$\text{for all } x_1, x_2 \in I, \ \ x_1 < x_2 \implies f(x_1) \le f(x_2)$$

be decided? If so, is it true or false?

♣ 5. Repeat (4) with $I = (0,4)$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $f(1) = 1$, $f(2) = 2$, $f(3) = 3$.

♣ 6. Sketch the graph of a function that is nondecreasing on the interval $(1,3)$, but *not* increasing on $(1,3)$.

♣ 7. Sketch the graph of a function f that is increasing on $(0,1)$ and nondecreasing on $(1,2)$. Is f nondecreasing on $(0,2)$? Justify your answer.

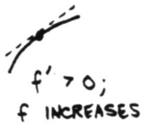
♣ 8. True or False: Every function that is increasing on I is nondecreasing on I.

True or False: Every function that is nondecreasing on I is increasing on I.

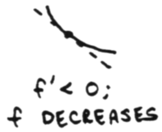
♣ 9. Based on your experience with nondecreasing functions, write down a precise definition of a *nonincreasing* function on an interval I.

getting increasing/ decreasing info from the derivative

If f is *differentiable* on (a, b), then for any $c \in (a, b)$, the number $f'(c)$ exists and tells us how fast the function values $f(x)$ are changing with respect to x at the point $(c, f(c))$. It seems plausible that this derivative information could be used to determine where f is increasing or decreasing: intuitively, where the slopes of the tangent lines are positive, the graph should be travelling UP (increasing), and where the slopes are negative, the graph should be travelling DOWN (decreasing).



It is indeed the case that increasing/decreasing information can be obtained from the sign of the derivative. The proof of this fact is a classic application of the Mean Value Theorem: using information about f' to glean information about f!



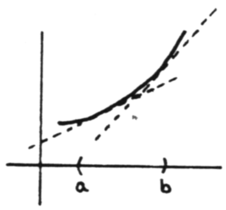
THEOREM

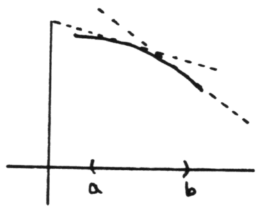
increasing, decreasing info from f'

Suppose that f is differentiable on (a, b).

If $f'(x) > 0$ for all $x \in (a, b)$, then f is *increasing* on (a, b).

If $f'(x) < 0$ for all $x \in (a, b)$, then f is *decreasing* on (a, b).





At first glance, one might be concerned that the Mean Value Theorem does not apply here, since the requirement about continuity at the endpoints of the interval has not been met. Make sure you understand how this 'problem' is circumvented in the following proof.

PARTIAL PROOF

Proof. Let f be differentiable on (a, b) and suppose that $f'(x) > 0 \ \forall x \in (a, b)$. Choose *any* x_1, x_2 in (a, b) with $x_1 < x_2$ (so that $x_2 - x_1 > 0$). Observe that x_1 cannot be a, since $a \notin (a, b)$. Similarly, x_2 cannot be b.

Since f is differentiable at x_1 and x_2 (by hypothesis), f must also be continuous at x_1 and x_2. (♣ Why?) Thus, f is not only differentiable on the open interval (x_1, x_2), but also continuous on the closed interval $[x_1, x_2]$. Thus, the Mean Value Theorem guarantees existence of a number c in (x_1, x_2) for which:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But since $c \in (a, b)$ and $f'(x) > 0 \ \forall x \in (a, b)$, we have $f'(c) > 0$. Thus:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

Multiplying both sides of this inequality by the positive number $x_2 - x_1$ yields the equivalent inequality

$$f(x_2) - f(x_1) > 0 \ ,$$

that is, $f(x_2) > f(x_1)$. It has been shown that *whenever* $x_1, x_2 \in I$ satisfy $x_1 < x_2$, it is also true that $f(x_1) < f(x_2)$. So, f is increasing on I.

The remaining case is left as an exercise. ∎

EXERCISE 4

proof of the remaining case

♣ Prove the following result:

If f is differentiable on (a, b) and $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on (a, b).

Use the previous proof as a guide, and make appropriate changes. Be sure to write complete mathematical sentences.

shorter forms of the proof

More advanced students of mathematics would condense the proof a bit. Here's what a shorter proof might look like:

Proof. Let f be differentiable on (a, b) with $f'(x) > 0 \ \forall x \in (a, b)$. Choose any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Since f is differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$, the MVT guarantees existence of a number $c \in (x_1, x_2)$ for which:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But $f'(c) > 0$ yields the desired conclusion that $x_1 < x_2 \implies f(x_1) < f(x_2)$. ∎

EXERCISE 5

write a shorter proof

♣ Prove that if f is differentiable on (a, b) and $f'(x) \geq 0$ for all $x \in (a, b)$, then f is nondecreasing on (a, b). Use the previous 'shorter proof' as a guide, and make appropriate changes. Be sure to write complete mathematical sentences.

shortest proof

A real expert would merely say the following:

Proof. The proof is a direct consequence of the Mean Value Theorem. ∎

EXAMPLE

Problem: Consider the function:

$$P(x) = 2x^3 + 3x^2 - 12x$$

Find the open intervals on which P increases; decreases.

First Solution: Differentiation yields:

$$\begin{aligned} P'(x) &= 6x^2 + 6x - 12 \\ &= 6(x^2 + x - 2) \\ &= 6(x-1)(x+2) \end{aligned}$$

Where is $P'(x)$ positive? Negative? Recall that a product ab of real numbers is positive in two situations:

$$ab > 0 \quad \iff \quad (a > 0 \text{ and } b > 0) \text{ or } (a < 0 \text{ and } b < 0)$$

EXERCISE 6
investigating $ab > 0$

♣ 1. What *are* the two situations for which ab is positive? Answer in English.

Remember that the symbol '$\iff$' in the sentence

$$ab > 0 \quad \iff \quad (a > 0 \text{ and } b > 0) \text{ or } (a < 0 \text{ and } b < 0)$$

tells us that the 'smaller' sentences being compared *always* have the same truth values. If one is true, so is the other; if one is false, so is the other.

♣ 2. If $a = 1$ and $b = 2$, is the sentence '$ab > 0$' true or false? How about the sentence '$(a > 0$ and $b > 0)$ or $(a < 0$ and $b < 0)$'?

♣ 3. Repeat (2), taking $a = 1$ and $b = -2$.

Returning to the example, we now investigate where $P'(x)$ is positive:

$$\begin{aligned} \{x \mid P'(x) > 0\} &= \{x \mid x - 1 > 0 \text{ and } x + 2 > 0\} \cup \{x \mid x - 1 < 0 \text{ and } x + 2 < 0\} \\ &= \{x \mid x > 1 \text{ and } x > -2\} \cup \{x \mid x < 1 \text{ and } x < -2\} \\ &= \{x \mid x > 1\} \cup \{x \mid x < -2\} \\ &= (1, \infty) \cup (-\infty, -2) \end{aligned}$$

Thus, $P'(x)$ is positive on $(-\infty, -2) \cup (1, \infty)$, so P increases on these intervals.

EXERCISE 7

♣ 1. What happened to the number 6 that appears in the formula for $P'(x)$?

♣ 2. One line in the previous display used the fact that:

$$x > 1 \text{ and } x > -2 \quad \iff \quad x > 1$$

Where was this fact used? Convince yourself that it is indeed true.

EXERCISE 8

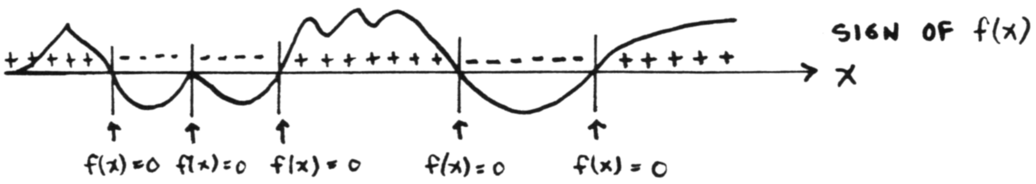
♣ Now use the fact that

$$ab < 0 \quad \iff \quad (a < 0 \text{ and } b > 0) \text{ or } (a > 0 \text{ and } b < 0)$$

to find the open intervals on which $P(x) = 2x^3 + 3x^2 - 12x$ decreases. Be sure to write complete mathematical sentences.

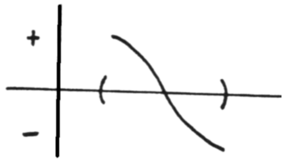
a better approach

There is a much easier approach to the previous problem, that exploits the *continuity* of P'. The places where a *continuous* function is positive and negative can be easily determined, merely by finding out where the function is *zero*, and then testing some points in between! This technique is discussed next.



a useful consequence of the Intermediate Value Theorem

If a function f is continuous on an interval and takes on *both* positive and negative values on this interval, then it must also take on the value 0. This fact is an immediate consequence of the Intermediate Value Theorem. (♣ Why?)



So suppose that f is continuous on I, and consider the implication:

IF f takes on both positive and negative values on I,
THEN f takes on the value 0 on I.

The contrapositive of this implication is:

IF f is nonzero on I,
THEN f is either positive or negative on I.

Since an implication is equivalent to its contrapositive, it has been shown that if a function is *continuous* and *nonzero* on an interval, then it must be *either positive or negative on this interval.*

EXERCISE 9

♣ 1. Convince yourself that:

$$\text{not}(f \text{ takes on the value } 0 \text{ on } I) \quad \iff \quad f \text{ is nonzero on } I$$

♣ 2. Convince yourself that:

$$\begin{aligned}&\text{not}(f \text{ takes on both positive and negative values on } I)\\ &\quad \iff \quad f \text{ is either positive or negative on } I\end{aligned}$$

finding out where an arbitrary function is positive and negative; the 'number line' approach

Now, to find out where an *arbitrary* (not necessarily continuous) function is positive or negative, proceed as follows:

- Find all the discontinuities; mark these on a number line.
- Find all the places where the function is zero; mark these on the number line.
- The function must be continuous and nonzero on every subinterval. So, it must be either positive or negative on these subintervals!
- Choose a test point in each subinterval to determine the sign (+ or −) of the function there.

For lack of a better name, this will be called the 'number line' approach.

The 'number line' approach is used next to find where the function $P(x) = 20x^2 + 8x - 1$ is positive and negative.

EXAMPLE

using the 'number line' approach

Problem: Use the 'number line approach' to find where $P(x) = 20x^2 + 8x - 1$ is positive and negative.

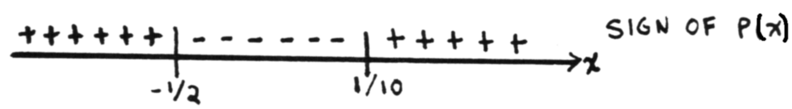
Solution: The function P is continuous everywhere. To determine where it is zero, either use the quadratic formula, or factor:

$$P(x) = (10x - 1)(2x + 1)$$

Then:

$$\begin{aligned} P(x) = 0 \quad &\iff \quad (10x-1)(2x+1) = 0 \\ &\iff \quad 10x - 1 = 0 \ \text{ or } \ 2x + 1 = 0 \\ &\iff \quad x = \frac{1}{10} \ \text{ or } \ x = -\frac{1}{2} \end{aligned}$$

Mark these zeroes on a number line. It is nice to label the number line as '*SIGN OF* $P(x)$'.



There are three subintervals formed; on each, P is continuous and nonzero. A test point must be chosen from each subinterval to determine the sign of $P(x)$ there. Choose the simplest test point to work with.

When $x = -1$: $P(-1) = (10(-1) - 1)(2(-1) + 1) = (-11)(-1) > 0$

When $x = 0$: $P(0) = (10(0) - 1)(2(0) + 1) = (-1)(1) < 0$

When $x = 1$: $P(1) = (10(1) - 1)(2(1) + 1) = (9)(3) > 0$

Note that, in all three cases, it was not necessary to determine the exact function values. Our only interest is the SIGN of the result.

In conclusion, $P(x)$ is positive on $(-\infty, -\frac{1}{2}) \cup (\frac{1}{10}, \infty)$ and negative on $(-\frac{1}{2}, \frac{1}{10})$.

using the 'number line' approach to determine where a function increases, decreases.

To simplify the problem of determining where a function is increasing and decreasing, just apply the 'number line' approach to the first derivative of the function, as in the next example.

EXAMPLE

Problem: Consider the function from an earlier example:

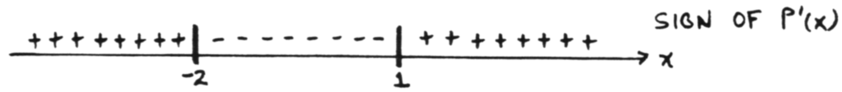
$$P(x) = 2x^3 + 3x^2 - 12x$$

Find the open intervals on which P increases; decreases.

Solution: The earlier example showed that P has derivative:

$$P'(x) = 6(x-1)(x+2)$$

This derivative P' is continuous everywhere; it is zero at $x = 1$ and $x = -2$. Plot these points on a number line, as shown below. It is nice to label the number line as '*SIGN OF* $P'(x)$'.



There are three subintervals formed, and a test point must be chosen from each.

When $x = -3$: $P'(-3) = 6(-3-1)(-3+2) = 6(-4)(-1) > 0$

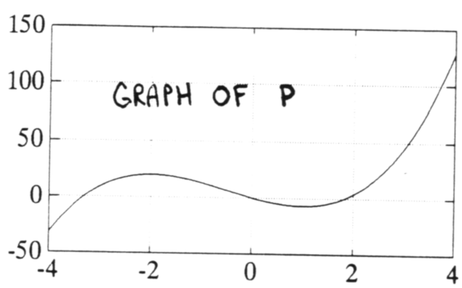
When $x = 0$: $P'(0) = 6(0-1)(0+2) = 6(-1)(2) < 0$

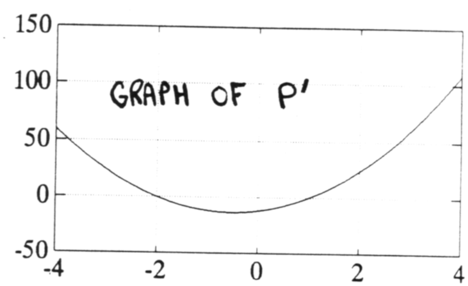
When $x = 2$: $P'(2) = 6(2-1)(2+2) = 6(1)(4) > 0$

Thus, $P'(x)$ is positive on $(-\infty, -2) \cup (1, \infty)$ and negative on $(-2, 1)$.

Thus, P increases on $(-\infty, -2) \cup (1, \infty)$ and decreases on $(-2, 1)$. This is (of course) in agreement with earlier results.

The graphs of P and P' are shown below.





The next problem is written in a very abbreviated form. Feel free to use this form when working the End-Of-Section exercises.

EXAMPLE

the 'number line' approach with a discontinuous function; abbreviated form of solution

Problem: Find the open intervals on which the function $f(x) = \frac{x^2}{x-1}$ increases and decreases.

Solution:

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

Discontinuity: at $x = 1$

Zeroes of f':

$$\begin{aligned} f'(x) = 0 \quad &\iff \quad x(x-2) = 0 \\ &\iff \quad x = 0 \ \text{ or } \ x = 2 \end{aligned}$$

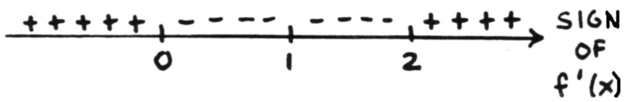
Test Points:

$-1:\quad \frac{(-1)(-1-2)}{+} = \frac{(-)(-)}{+} > 0$

$\frac{1}{2}:\quad \frac{(\frac{1}{2})(\frac{1}{2}-2)}{+} = \frac{(+)(-)}{+} < 0$

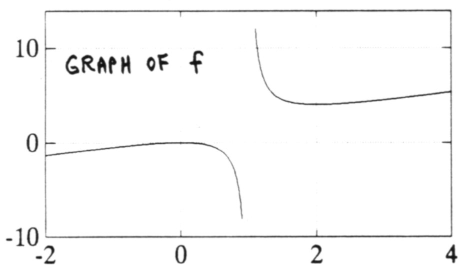
$\frac{3}{2}:\quad \frac{(\frac{3}{2})(\frac{3}{2}-2)}{+} = \frac{(+)(-)}{+} < 0$

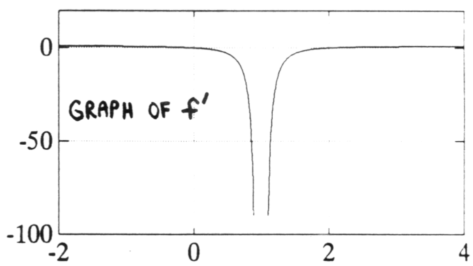
$3:\quad \frac{3(3-2)}{+} = \frac{(+)(+)}{+} > 0$



Conclusion: f increases on $(-\infty, 0) \cup (2, \infty)$ and decreases on $(0, 1) \cup (1, 2)$.

The graphs of f and f' are shown below.





QUICK QUIZ

sample questions

♣ 1. Give a precise definition of what it means for a function f to increase on an interval I.

♣ 2. Find the open interval(s) on which $f(x) = x(x-1)$ is positive and negative. Use the 'number line' approach.

♣ 3. True or False: If f is nonzero and continuous on an interval I, then f is either positive or negative on I.

♣ 4. True or False: If $f'(x) > 0$ for all x in an interval I, then f increases on I.

KEYWORDS

for this section

Increasing and decreasing functions—precise definitions, nonincreasing and nondecreasing functions, getting inc/dec information from the derivative, using the Intermediate Value Theorem to decide where a function is positive and negative.

END-OF-SECTION EXERCISES

Find the open intervals on which the functions are positive and negative. Use any appropriate method. In many cases, the 'number line' approach may be easiest.

1. $P(x) = x^2 + x - 2$
2. $P(x) = x^2 - x - 2$
3. $P(x) = 2x^2 - 4x - 6$
4. $P(x) = 3x^2 + 6x - 9$
5. $P(x) = 12x^2 - 13x + 3$
6. $P(x) = 14x^2 + 3x - 2$
7. $P(x) = x^3 + 2x^2 + x$
8. $P(x) = x^3 - 2x^2 + x$

9 $P(x) = x^3 + 4x^2 - x - 4$ Hint: $x = 1$ is a zero of P

10 $P(x) = x^3 - 13x - 12$ Hint: $x = -1$ is a zero of P

11. $f(x) = x^2e^x - x^2$
12. $f(x) = e^x x^2 - e^x$

13 $f(x) = \ln(2x - 1)$ Hint: $\mathcal{D}(f) = (\frac{1}{2}, \infty)$, so the 'number line' approach is applied to a 'partial' number line. Also: $\ln(2x - 1) = 0 \iff 2x - 1 = 1$

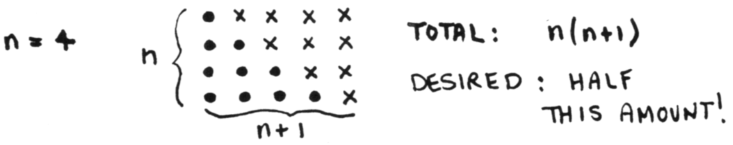
14 $f(x) = \ln(1 - 2x)$ Hint: $\mathcal{D}(f) = (-\infty, \frac{1}{2})$, and $\ln(1 - 2x) = 0 \iff 1 - 2x = 1$

Find the open intervals on which the functions increase and decrease. Use any appropriate method. In many cases, the 'number line' approach applied to the first derivative may be easiest.

15. $f(x) = 2x^3 + 3x^2 - 12x + 1$
16. $f(x) = x^3 - 3x^2 - 9x + 4$
17. $f(x) = xe^x$
18. $f(x) = x^2e^x$
19. $f(x) = x \ln x$
20. $f(x) = x^2 \ln x$

21 (formula for finding $1 + 2 + \cdots + n$) The picture below gives a geometric proof that:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$



a. Explain how the formula 'comes from' the picture.

b. Find: $1 + 2 + \cdots + 67$

c. Find: $S := 64 + 65 + \cdots + 108$ Hint: $S = (1 + 2 + \cdots + 108) - (1 + 2 + \cdots + 63)$

22 (refer to (21)) It was seen in the previous problem that the function $S\colon \{1, 2, 3, \dots\} \to \mathbb{R}$ defined by $S(n) = \frac{n(n+1)}{2}$ gives as its outputs the sum of the first n positive integers.

As n increases, so does $S(n)$. (♣ Why?) To check this, first define a function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x(x+1)}{2}$, (the 'analogue' of S that is defined for all real numbers, not only the positive integers).

a. If it can be shown that f increases on an interval that contains the positive integers, then S must also be increasing. Convince yourself that this is true.

b. Show that f increases on $(-\frac{1}{2}, \infty)$. Conclude that S is an increasing function.

END-OF-SECTION EXERCISES

(continued)

23 For all real numbers x except 1, and for all positive integers n:

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

Here is how the formula is derived:

a. Define: $S := 1 + x + \cdots + x^n$

Find: $x \cdot S$

b. Show that: $xS - S = x^{n+1} - 1$

Conclude that the formula for S has the desired form.

c. Use the formula for S to find $1+2+2^2+2^3+2^4$. Verify that your result is correct, by performing the indicated additions and exponentiations.

d. Use the formula for S to find $2^6 + 2^7 + 2^8 + 2^9 + 2^{10}$. Again, verify that your result is correct.

24 (refer to (23)) As in the previous example, define:

$$S(x) = 1 + x + x^2 + \cdots + x^n$$

a. For a fixed value of n and for positive x, one would suspect that as x increases, so does $S(x)$. Comment on why this is true.

b. Let $n = 2$ in the formula for $S(x)$. Determine where S increases and decreases. Think about your result.

c. Let $n = 3$ in the formula for $S(x)$. Determine where S increases and decreases. Think about your result.

25 (Probability). This problem shows a situation in probability that gives rise to sums of the form $1 + x + \cdots + x^n$.

On a fair dice (6 sides), the probability of getting at least one ‘2’ in the first n throws is given by:

$$P(n) = \frac{1}{6}[1 + \frac{5}{6} + (\frac{5}{6})^2 + \cdots + (\frac{5}{6})^{n-1}]$$

To get a feeling for where this formula comes from, let’s check a couple cases:

a. Let $n = 1$. Discuss why the probability of getting a ‘2’ on one throw is $\frac{1}{6}$. Verify that the formula is correct in this case.

b. Let $n = 2$. How can we get at least one ‘2’ in two throws? We could get a 2 on the first throw; the probability of this is $\frac{1}{6}$. Or, we could get a number other than ‘2’ on the first throw, and a ‘2’ on the second throw; the probability of this is $(\frac{5}{6})(\frac{1}{6})$. The total probability is $\frac{1}{6} + (\frac{5}{6})(\frac{1}{6})$. Verify that the formula is correct in this case.

c. Conjecture that as n increases, so should $P(n)$. Why? Find $P(1)$, $P(2)$ and $P(3)$. Are these results in agreement with your conjecture?

d. Prove that, for all positive integers n, $P(n) < P(n + 1)$. Thus, P is indeed an increasing function. Hint: Compare the formulas for $P(n)$ and $P(n + 1)$. What do they differ by? Is the difference always positive?