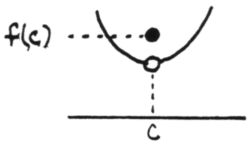
5.2 Local Maxima And Minima
Critical Points

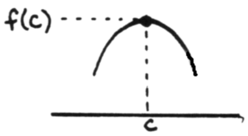
Introduction

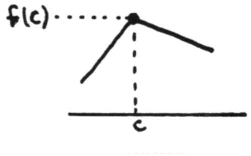
Often, one is interested in *maximizing* or *minimizing* functions; say, maximizing profits or minimizing costs. In the section on the Max-Min Theorem, maximum and minimum values of a function on an interval were discussed. Now, this idea is extended so that we can talk about *local maximum and minimum values for a function.*

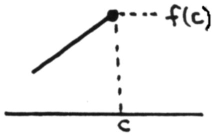
local maximum;
local minimum;
informal discussion

A number $f(c)$ is called a *local maximum* for a function f, if, *locally*, it's the *highest value*. That is, for all x sufficiently close to c, it must be that $f(x) \leq f(c)$. The sketches below illustrate some of the ways that this can happen.







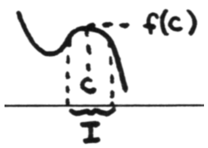


Analogously, a number $f(c)$ is called a *local minimum* for a function f, if, *locally*, it's the *lowest value.*

EXERCISE 1	♣ Make four sketches that illustrate different ways a function f may have a local minimum.

Here are the precise definitions of *local maximum* and *local minimum*:

DEFINITION
local maximum,
local minimum,
local extreme values



Let f be a function and let $c \in \mathcal{D}(f)$.

The number $f(c)$ is a *local maximum for* f if and only if there exists $\delta > 0$ such that whenever $|x - c| < \delta$ and $x \in \mathcal{D}(f)$, then $f(x) \leq f(c)$.

The number $f(c)$ is a *local minimum for* f if and only if there exists $\delta > 0$ such that whenever $|x - c| < \delta$ and $x \in \mathcal{D}(f)$, then $f(x) \geq f(c)$.

When such maximum or minimum values occur, they are called *local extreme values of* f.

a slight
abuse of notation

Strictly speaking, the *number* $f(c)$ is the local extreme value for f. However, one often abuses notation and speaks of, say, the *local maximum (point)* $(c, f(c))$. This is because one is *not only interested in the local maximum value* $f(c)$*, but also the place* c *where it occurs.*

EXERCISE 2

Sketch the graph of a function satisfying:

♣ 1. f has a local maximum at $x = 2$, $f(2) = 4$, f is not differentiable at $x = 2$

♣ 2. $\mathcal{D}(f) = [1, 2]$, f has a local minimum at $x = 1$, $f(2) < f(1)$, the maximum value of f on $[1, 2]$ is 5

♣ 3. f has a local maximum at $x = 1$, $f(1) = 2$, $\{x \mid f(x) \in (2, 3)\} = \emptyset$, f has a local minimum at $x = 2$

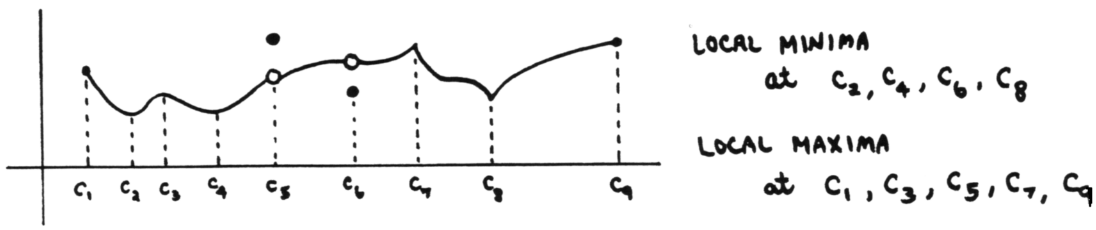
EXERCISE 3

♣ Suppose a function f has the following property: $f(2) = 3$, and arbitrarily close to $x = 2$ there are inputs whose function values are greater than 3. Can f have a local maximum at $x = 2$? Why or why not? How about a local minimum at $x = 2$?

a function may have many local extrema

The plural of *maximum* is *maxima*. The plural of *minimum* is *minima*. The singular of *extrema* is *extremum*.

Observe that a function may have *many* local extrema, as illustrated below.

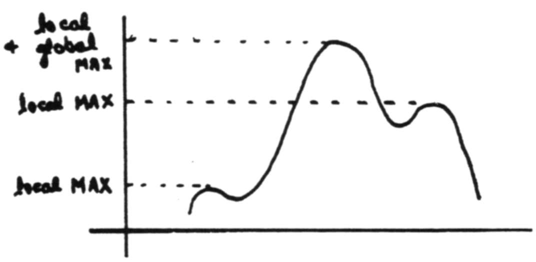


local versus global extrema

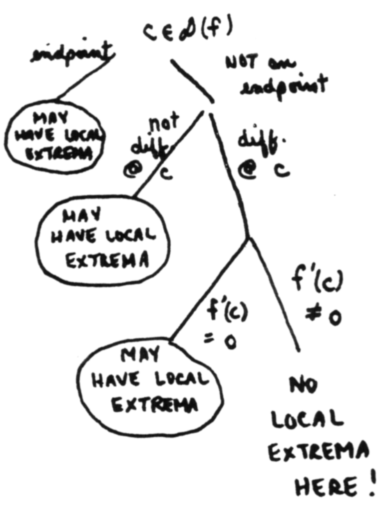
Occasionally, one is interested in the highest value that a function f attains on some specified set (like the highest value f attains on an interval; or the highest value f attains over its entire domain). Such a highest value (if it exists) is called a *global maximum*.

Similarly, one can speak of a *global minimum*.

Right now, we won't concern ourselves with *global* extrema. However, later on we'll return to this idea. For the moment, just be aware that the adjective *local* is used to distinguish the discussion from the *global* case.

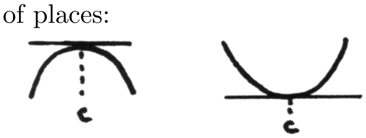


finding local maxima and minima



It appears, from the examples put forth thus far, that local maxima and minima seem to occur at certain types of places:

(a) places where $f'(c) = 0$



(b) endpoints of a domain

(c) places where the derivative does not exist

Indeed, *these are the ONLY types of places where local extreme values can occur.* This fact is proven in the next few paragraphs. Refer to the flow chart at left as you read through the following argument.

proof that local extreme values can only occur at certain types of places

Let c be in the domain of f.

Either c is an endpoint of the domain of f, or not. If it is, then there *may* be a local extreme value at c, as in situation (b) above.

Now suppose that c is not an endpoint. Either f is differentiable at c or not. If not, then there *may* be a local extreme value at c, as in case (c) above.

Now suppose f *is* differentiable at c. Then either $f'(c) \neq 0$, or $f'(c) = 0$. If $f'(c) = 0$, then there *may* be a local extreme value at c, as in case (a) above.

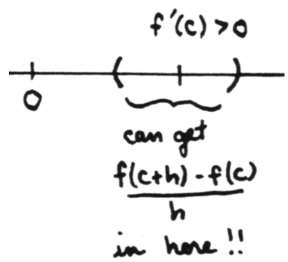
IF: $c \in \mathcal{D}(f)$, c is not an endpoint, f is differentiable at c, $f'(c) \neq 0$; then $f(c)$ is NOT a local extreme value for f

All that is needed to complete the proof is to show the following:

IF:

- $c \in \mathcal{D}(f)$,
- c is not an endpoint of the domain of f,
- f is differentiable at c, and
- $f'(c) \neq 0$,

THEN the number $f(c)$ *is not* a local extreme value for f.



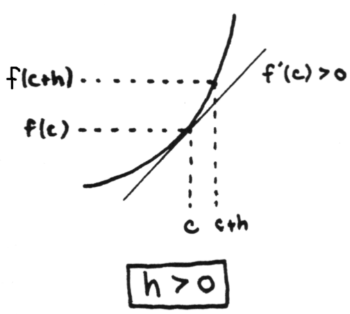
So assume all these things are true. Since $f'(c) \neq 0$, it must be that either $f'(c) < 0$ or $f'(c) > 0$. Suppose for the moment that $f'(c) > 0$. (You'll investigate the case $f'(c) < 0$ in the exercises.) That is:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} > 0$$

This means that *we can get the values* $\frac{f(c+h)-f(c)}{h}$ *as close to the number* $f'(c)$ *as desired*, merely by restricting ourselves to values of h sufficiently close to 0.

So choose δ so small that whenever $|h - 0| < \delta$, one has:

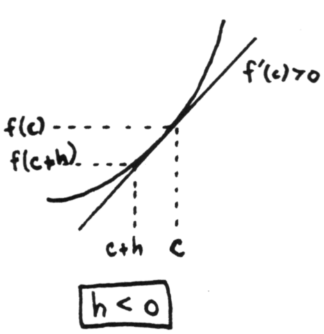
$$\frac{f(c+h) - f(c)}{h} > 0 \qquad (*)$$



By hypothesis, c is *not* an endpoint of the domain of f, so the function is defined both to the right and left of c. Now, whenever $h > 0$ and $|h| < \delta$, multiplying both sides of (*) by the positive number h yields the equivalent inequality

$$f(c+h) - f(c) > 0 \ ,$$

so that $f(c+h) > f(c)$. That is, *arbitrarily close to $(c, f(c))$, on the right, is another point $(c+h, f(c+h))$ with a greater function value, $f(c+h) > f(c)$.* Thus, $f(c)$ is NOT a local maximum.



Similarly, whenever $h < 0$ and $|h| < \delta$, multiplying both sides of (*) by the negative number h yields the equivalent inequality

$$f(c+h) - f(c) < 0 \ ,$$

so that $f(c+h) < f(c)$. That is, *arbitrarily close to $(c, f(c))$, on the left, is another point $(c+h, f(c+h))$ with a lesser function value.* Thus, $f(c)$ is NOT a local minimum, either.

Thus, f does not have a local extreme value at $x = c$.

EXERCISE 4

♣ Prove the following:

If $c \in \mathcal{D}(f)$, c is not an endpoint of the domain of f, f is differentiable at c, and $f'(c) < 0$, then the number $f(c)$ is NOT a local extreme value for f.

What has been learned?

CRITICAL POINTS;

candidates for places where local extrema occur

Summarizing:

IF a function f has a local extremum at the point $(c, f(c))$, THEN

- c is an endpoint of the domain of f; or
- f is *not* differentiable at c; or
- $f'(c) = 0$

Any point that satisfies at least one of these three conditions is called a *CRITICAL POINT.* The critical points are the CANDIDATES for places where local extrema can occur.

EXERCISE 5

more practice with implications

The sentences $P \Rightarrow Q$ and $Q \Rightarrow P$ are *not equivalent*, as the truth table below shows.

P	Q	P ⇒ Q	Q ⇒ P
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

♣ 1. What truth values of P and Q make $P \Rightarrow Q$ true, but $Q \Rightarrow P$ false?

♣ 2. What truth values of P and Q make $Q \Rightarrow P$ true, but $P \Rightarrow Q$ false?

♣ 3. What truth values of P and Q make both $P \Rightarrow Q$ and $Q \Rightarrow P$ true?

In particular, *even if it is known that* $P \Rightarrow Q$ *is true*, it is NOT possible to conclude that $Q \Rightarrow P$ must also be true.

♣ 4. Argue that the implication $x = 2 \implies x^2 = 4$ is true. That is, show that for *all* values of x, the sentence is true.

♣ 5. Argue that the implication $x^2 = 4 \implies x = 2$ is false. That is, show that there is *at least one value of* x for which the sentence is false.

CAUTION:

a critical point MAY or MAY NOT be a place where a local extreme value occurs

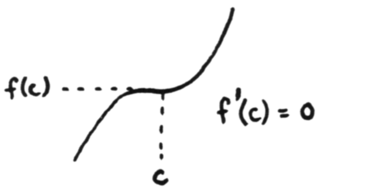
It has been proven that the following sentence is always true:

IF f has a local extreme value at the point $(c, f(c))$,
THEN the point $(c, f(c))$ is a critical point.

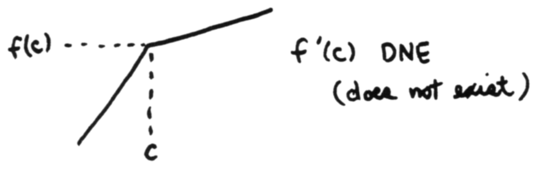
The previous exercise points out that this knowledge *DOES NOT* determine the truth of the sentence:

IF the point $(c, f(c))$ is a critical point,
THEN f has a local extreme value at $(c, f(c))$.

Indeed, the sketches below indicate that there *are* critical points that *do not* correspond to local extreme values.



(c, f(c)) is a CRITICAL PT., but not a local extremum



(c, f(c)) is a CRITICAL PT., but NOT a local extremum

EXAMPLE

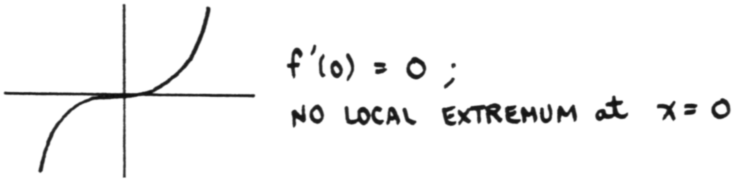
Problem: Determine if the following sentences are TRUE or FALSE. If false, make a sketch showing that it is false.

- *If f has a local minimum at $x = 2$, then the point $(2, f(2))$ is a critical point for f.*

Solution: True! Any local extrema MUST OCCUR at a critical point.

- *If $f'(c) = 0$, then f must have a local maximum or minimum at c.*

Solution: False! The function $f(x) = x^3$ satisfies $f'(0) = 0$. However, f does not have a local extremum at $x = 0$.



EXERCISE 6

Determine if the following sentences are TRUE or FALSE. If false, make a sketch showing that it is false.

♣ 1. If f has a local maximum at $x = 4$, then the point $(4, f(4))$ must be a critical point for f.

♣ 2. If f has a local maximum at $x = 4$, then it must be that $f'(4) = 0$.

♣ 3. If $\mathcal{D}(f') = \mathbb{R}$ and f has a local maximum at $x = 4$, then it must be that $f'(4) = 0$.

♣ 4. If $f'(1) = 0$, then $(1, f(1))$ is a local extreme point for f.

strategy for finding local extrema

Given a function f, here is a strategy for finding all the local extrema of f:

First, find *all places* in $\mathcal{D}(f)$ where the derivative is zero, where the derivative doesn't exist, and all the endpoints of the domain of f. That is, find all the critical points for f.

Then, investigate each of these critical points to see if it *is* or *is not* a local extremum for f.

EXAMPLE

Problem: Find all local extreme values for the function $f\colon [0,5] \to \mathbb{R}$, $f(x) = (x-2)^2$.

Solution:

- Note that $\mathcal{D}(f) = [0,5]$. All critical points must come from the domain of f.
- The endpoints of $\mathcal{D}(f)$ are critical points. Calculate the function values at the endpoints:

$$\begin{aligned} f(0) &= (0-2)^2 = 4 \\ f(5) &= (5-2)^2 = 9 \end{aligned}$$

  Thus, $(0,4)$ and $(5,9)$ are critical points.
- Find f'.

$$f'(x) = 2(x-2)(1) = 2(x-2)\ .$$

- Find all places where $f'(x) = 0$. Be sure to write a complete mathematical sentence!

$$\begin{aligned} f'(x) = 0 &\iff 2(x-2) = 0 \\ &\iff x = 2 \end{aligned}$$

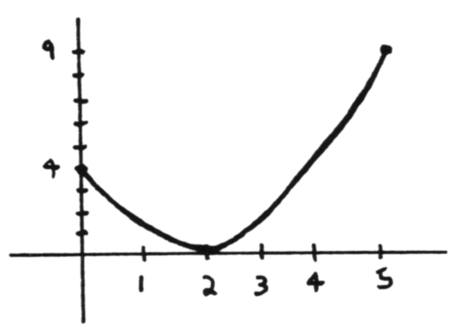
  Note that $f(2) = (2-2)^2 = 0$. So, $(2,0)$ is a critical point.
- There are no places where $f'(c)$ does not exist.

It is convenient to summarize all this information by constructing a table:

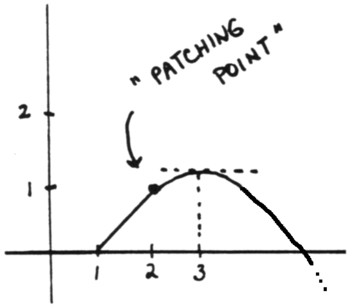
C	f(c)	WHY A CRIT. PT?	LOCAL EXTREMUM?
0	4	ENDPT OF DOMAIN	MAX
5	9	ENDPT OF DOMAIN	MAX
2	0	$f'(c) = 0$	MIN

Now, IF f has any local extrema, they must occur at these three points.

Indeed, a quick sketch of the graph of f is easy to get in this case: take the graph of $y = x^2$, and shift it two units to the right. Thus, we see that there are local maxima at $(0,4)$ and $(5,9)$, and a local minimum at $(2,0)$.



EXAMPLE



Problem: Find all local extreme values for the function $f\colon [1,\infty) \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} x-1 & \text{for } x \in [1,2] \\ -\frac{1}{8}(x^2-6x) & \text{for } x > 2 \end{cases}$$

Observe that f is continuous at the point $(2,1)$. This is because

$$\lim_{x\to 2^-} f(x) = \lim_{x \to 2^-} x - 1 = 2 - 1 = 1$$

and:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} -\frac{1}{8}(x^2-6x) = -\frac{1}{8}(2^2 - 6\cdot 2) = 1$$

Clearly, f is differentiable everywhere, except possibly at the 'patching point' $(2,1)$, and:

$$f'(x) = \begin{cases} 1 & \text{for } x \in [1,2) \\ -\frac{1}{4}(x-3) & \text{for } x > 2 \end{cases}$$

Is f differentiable at $(2,1)$? It is possible, in this case, to answer without computing a difference quotient. We need only look at the formulas for f', and think.

What is the 'direction' at $(2,1)$, coming in from the left? Here, the graph is a line with slope 1, so the 'direction' coming in from the left is 1.

What is the 'direction' at $(2,1)$, coming in from the right? Arbitrarily close to $(2,1)$, on the right, the tangent lines are given by the function $-\frac{1}{4}(x-3)$. And as x approaches 2, this function approaches $-\frac{1}{4}(2-3) = \frac{1}{4}$. Thus, the 'direction' coming in from the right is $\frac{1}{4}$.

Since $1 \neq \frac{1}{4}$, f is not differentiable at the point $(2,1)$.

★★ The precise justification for the argument above is really quite subtle. It uses the continuity of f at and near c, the continuity of f' near c, and an interchange of limit operations. However, it should be intuitive to students that this is possible.

Finally:

$$\begin{aligned} f'(x) = 0 \quad &\iff \quad -\frac{1}{4}(x-3) = 0 \\ &\iff \quad x = 3 \end{aligned}$$

Note that $f(3) = -\frac{1}{8}(3^2 - 6\cdot 3) = \frac{9}{8}$. Since the graph of any function of the form $f(x) = ax^2 + bx + c$ (for $a \neq 0$) is known to be a parabola, it is now possible to make a quick sketch of the function.

If f has any local extrema, they must occur at the places listed below:

c	f(c)	WHY A CRIT PT?	LOCAL EXTREMUM?
1	0	ENDPT of DOMAIN	MIN
2	1	f'(c) DNE	NO
3	9/8	f'(c) = 0	MAX

From the sketch, it is easy to see that $(1,0)$ is a local minimum, $(2,1)$ is not a local extremum, and $(3, \frac{9}{8})$ is a local maximum.

EXERCISE 7

Find all local extrema of the following functions, by finding and checking all the critical points. In each case, sketch the graph of the function.

♣ 1. $f\colon [1,\infty) \to \mathbb{R}$, $f(x) = (x-4)^3$

♣ 2. $f\colon [-1,4] \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} -x^2+1 & \text{for } x \in [-1,1] \\ x-1 & \text{for } x \in (1,4] \end{cases}$$

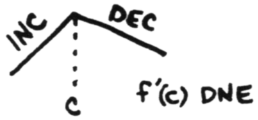
♣ 3. $f\colon [-1,4] \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} -x^2+1 & \text{for } x \in [-1,1] \\ -x+1 & \text{for } x \in (1,4] \end{cases}$$

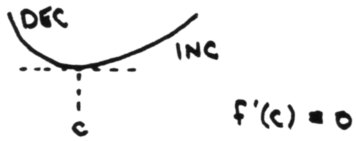
using information from f' to help determine local extreme behavior

Given a critical point, it may be difficult to determine if it corresponds to a local maximum, a local minimum, or neither, particularly when the graph of the function is not easily available. In many cases, the information that f' gives can be used to help investigate local extreme behavior at critical points.

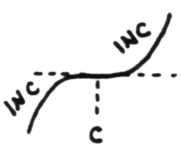
To see this, let $(c, f(c))$ be a critical point. Suppose that f is continuous at c and differentiable near c. If f increases to the left of c, and decreases to the right of c, then $f(c)$ must be a local maximum. That is, if $f'(x) > 0$ to the left of c, and $f'(x) < 0$ to the right of c, then $f(c)$ is a local maximum.

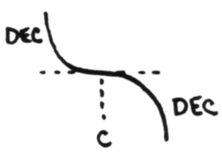


If f decreases to the left of c ($f'(x) < 0$), and increases to the right of c ($f'(x) > 0$), then $f(c)$ must be a local minimum.



If f increases both to the left and right of c, or if f decreases both to the left and right of c, then $f(c)$ is not a local extreme value.

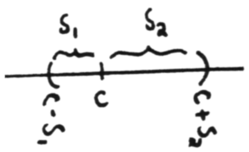




These observations are the content of a test that is commonly called the *First-Derivative Test*. It is stated precisely below.

FIRST DERIVATIVE TEST

for examining behavior at critical points



Suppose that $(c, f(c))$ is a critical point for f, so that f *may* have a local maximum or minimum at c.

Suppose that f is continuous at c. Also suppose that f is differentiable on $(c - \delta_1, c) \cup (c, c + \delta_2)$ for some positive numbers δ_1 and δ_2; that is, f is differentiable on intervals immediately to the left and right of c.

- If $f'(x) > 0$ on $(c - \delta_1, c)$ and $f'(x) < 0$ on $(c, c + \delta_2)$, then $(c, f(c))$ is a local maximum for f.
- If $f'(x) < 0$ on $(c - \delta_1, c)$ and $f'(x) > 0$ on $(c, c + \delta_2)$, then $(c, f(c))$ is a local minimum for f.
- If $f'(x)$ has the same sign on both intervals $(c - \delta_1, c)$ and $(c, c + \delta_2)$, then $(c, f(c))$ is neither a local maximum or minimum for f.

f must be continuous at c!

Note that if one takes away the hypothesis that f is continuous at c, then one is no longer guaranteed the conclusion of the theorem. To see this, consider the functions f sketched below. In both cases, f decreases to the left of c, and f increases to the right of c. However, the point $(c, f(c))$ is NOT a local minimum for the function f in either case.

EXERCISE 8

♣ 1. Sketch a function f satisfying: f increases to the left of c, decreases to the right of c, but $(c, f(c))$ is NOT a local maximum.

♣ 2. Sketch a function f satisfying: f increases to the left of c, decreases to the right of c, f is not continuous at c, and $(c, f(c))$ IS a local maximum.

'First-Derivative Tests' at endpoints

Similar results can be stated for endpoints. For example, suppose c is an endpoint of the domain of f, f is continuous at c, and $f'(x) > 0$ on some interval to the right of c. Then, $(c, f(c))$ must be a local minimum for f.

EXERCISE 9

♣ Precisely state a '*First-Derivative Test*' that applies to each of the three sketches shown below.

EXAMPLE

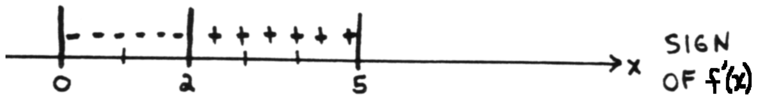
Problem: Reconsider an earlier example: Find all local extreme values for the function $f\colon [0,5] \to \mathbb{R}$, $f(x) = (x-2)^2$

There are three critical points, and first-derivative tests can be used to investigate each of these.

Since $f'(x) < 0$ to the right of $x = 0$, $(0,4)$ is a local maximum.

Since $f'(x) > 0$ to the left of $x = 5$, $(5,9)$ is a local maximum.

Since $f'(x) < 0$ to the left of $x = 2$, and $f'(x) > 0$ to the right of $x = 2$, $(2,0)$ is a local minimum.



EXERCISE 10

♣ Use first-derivative tests to investigate the critical points of an earlier example: the function $f\colon [1,\infty) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x-1 & \text{for } x \in [1,2] \\ -\frac{1}{8}(x^2-6x) & \text{for } x > 2 \end{cases}$$

EXAMPLE

Problem: Find all local extreme values for the function $f\colon [0,1) \cup (1,9] \to \mathbb{R}$ given by $f(x) = \frac{\sqrt{x}}{x-1}$.

Solution:

- Note that $\mathcal{D}(f) = [0,1) \cup (1,9]$. All critical points must come from the domain of f.
- The points $(0, f(0)) = (0,0)$ and $(9, f(9)) = (9, \frac{3}{8})$ are critical points, because they are endpoints of the domain of f.
- Find f' :

$$\begin{aligned} f'(x) &= \frac{(x-1)(\frac{1}{2}x^{-1/2}) - \sqrt{x}(1)}{(x-1)^2} \\ &= \ldots \quad \text{(simplify)} \\ &= \frac{-(x+1)}{2\sqrt{x}(x-1)^2} \end{aligned}$$

  The function f is defined at $x = 0$, but f' is not defined at 0, because this would produce division by 0. (Indeed, there is a vertical tangent line at $(0,0)$.)

- Find all places where $f'(x) = 0$:

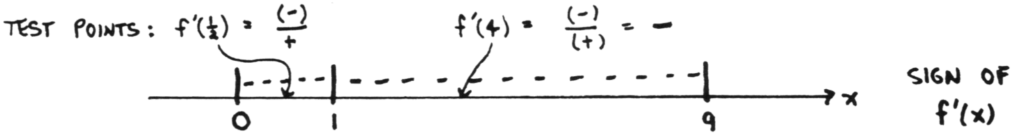
$$\begin{aligned} f'(x) = 0 \quad &\iff \quad -(x+1) = 0 \\ &\iff x = -1 \end{aligned}$$

  However, $x = -1$ is not in the domain of f.

This information is summarized below:

c	$f(c)$	WHY A CRIT. PT?	LOCAL EXTREMUM?
0	0	ENDPT of $\mathcal{D}(f)$, $f'(0)$ DNE	MAX
9	3/8	ENDPT OF $\mathcal{D}(f)$	MIN

To use first-derivative tests to investigate each critical point, the sign of $f'(x)$ must be determined. To do this, locate all places where $f'(x)$ is zero and discontinuous, and then use test points in each subinterval.



Conclude that $(0, 0)$ is a local maximum, and $(9, \frac{3}{8})$ is a local minimum.

Observe that it was NOT necessary to have the graph of f to determine all local maxima and minima. Later on, we will see a graph of this function.

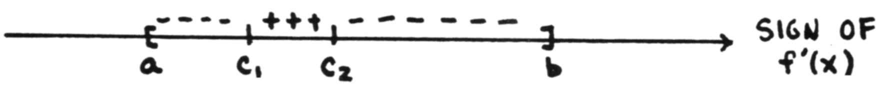
EXERCISE 11

♣ 1. Find all local extreme values for the function $f: [0, 1) \cup (1, 4) \to \mathbb{R}$ given by: $f(x) = \dfrac{e^x}{x-1}$

♣ 2. Find all local extreme values for the function $f: [0, 8] \to \mathbb{R}$ given by: $f(x) = \sqrt[3]{x} + x^3$

QUICK QUIZ
sample questions

1. Suppose that a function f has a local extremum at the point $(c, f(c))$. What (if anything) can be said about the behavior of f at this point?
2. Suppose that $(c, f(c))$ is a critical point for f. Must this point be a local maximum or a local minimum?
3. What is a 'critical point'?
4. Suppose it is known that '$A \implies B$' is true. Does this information alone determine the truth of '$B \implies A$'?
5. Suppose that a function $f: [a, b] \to \mathbb{R}$ is differentiable. The sign of $f'(x)$ is summarized below. Find all local extrema for f on $[a, b]$.



KEYWORDS
for this section

Local maxima and minima, local extreme values, local versus global extrema, critical points, strategy for finding local extrema, First Derivative Tests.

END-OF-SECTION EXERCISES

1. Find at least three sentences in this section that are *implicit* 'for all' sentences. That is, the words 'for all' do not appear, but are assumed to be there.

The problems below review concepts from sections 5.1 and 5.2.

♣ Classify each entry as an expression or a sentence.

♣ For any sentence, state whether it is TRUE or FALSE.

♣ All the implications are (either implicitly or explicitly) 'for all' sentences. If an implication is FALSE, give a COUNTEREXAMPLE. That is, provide a specific example where the hypothesis is TRUE, but the conclusion is FALSE.

2. If f increases on an interval I, then for all $a, b \in I$ with $a < b$, $f(a) < f(b)$
3. If f decreases on an interval I, then for all $a, b \in I$ with $a > b$, $f(a) < f(b)$
4. If g increases on an interval I, then for all $x, y \in I$, $g(x) \le g(y)$
5. If g decreases on an interval I, then for all $x, y \in I$ with $x < y$, $g(y) \le g(x)$
6. If $f(x) \le f(y)$ for all numbers x and y in an interval I with $x < y$, then f is nondecreasing on I.
7. If $g(a) \ge g(b)$ for all numbers a and b in an interval I with $a < b$, then g is nonincreasing on I.
8. If f if differentiable on (a, b) and $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on (a, b).
9. For all functions f, if f is increasing on an interval (a, b), then $f'(x) > 0$ on (a, b).
10. For all real numbers c and d:
$$cd > 0 \quad \Longleftrightarrow \quad (c > 0 \text{ and } d > 0) \text{ or } (c < 0) \text{ and } d < 0)$$
11. For all real numbers c and d:
$$cd < 0 \quad \Longleftrightarrow \quad (c > 0 \text{ and } d < 0) \text{ or } (c < 0 \text{ and } d > 0)$$
12. If f is continuous and nonzero on an interval I, then f is either positive or negative on I.
13. If f is continuous, and takes on both positive and negative values on an interval I, then f must also take on the value 0 on I.
14. If f has a local maximum at $(1, f(1))$, then the point $(1, f(1))$ is a critical point.
15. If f has a local minimum at $(2, f(2))$, then $f'(2) = 0$.
16. If $(c, f(c))$ is a critical point for f, then f has a local maximum or minimum at $(c, f(c))$.
17. If $\mathcal{D}(f) = \mathcal{D}(f') = \mathbb{R}$ and $(c, f(c))$ is a critical point for f, then $f'(c) = 0$.
18. If $P \Rightarrow Q$ is true, then $Q \Rightarrow P$ is true.
19. If f is continuous at c, $f'(x) > 0$ to the right of c, and $f'(x) < 0$ to the left of c, then the point $(c, f(c))$ is a local minimum for f.
20. If $f\colon [1, \infty) \to \mathbb{R}$ is continuous, and has the property that $f'(x) < 0$ for all $x \in (1, \frac{9}{8})$, then f has a local maximum at $x = 1$.