5.4 Graphing Functions
Some Basic Techniques

graphing a function of one variable

Given an arbitrary function of one variable, call it f, the *graph of* f is a 'picture' of the points $\{(x, f(x)) \,|\, x \in \mathcal{D}(f)\}$. Although the entire graph can rarely be shown (due to the fact that, say, $\mathcal{D}(f)$ is an infinite interval), one certainly wants to see everything interesting. These 'interesting' aspects usually include: local maxima and minima, global maxima and minima, inflection points, discontinuities, 'kinks', x and y-axis intercepts, asymptotes, and behavior at infinity.

Global maxima and minima are discussed in this section. Asymptotes and behavior at infinity are discussed in section 5.6.

Most of the tools necessary to take a systematic approach to graphing a function are now available. Some general guidelines are outlined below.

Graphing a function f

a systematic approach

Let f be a function of one variable. If the first two derivatives of f are reasonably easy to obtain, then the following strategy is suggested to obtain the graph of f:

- Find $\mathcal{D}(f)$, the domain of f. Sketch appropriate axes. Plot a few easy points. In particular, plot any endpoints of the domain of f.
- Note if the function is symmetric about the y axis or the origin. (See the Algebra Review in this section.) If so, the function only needs to be graphed for, say, nonnegative x, and the rest filled in from symmetry.
- Find $f'(x)$.

  Find all $c \in \mathcal{D}(f)$ where $f'(c) = 0$ or $f'(c)$ does not exist.

  Plot these points $(c, f(c))$ with the symbol ' $\times$ ' (if $f'(c)$ does not exist), or with the symbol '$-\ast-$' (if $f'(c) = 0$).

  These points, together with the endpoints of $\mathcal{D}(f)$, are the critical points. They are the *candidates* for local maxima and minima.
- Find $f''(x)$.

  Find all $c \in \mathcal{D}(f)$ where $f''(c) = 0$ or $f''(c)$ does not exist.

  Plot these points $(c, f(c))$ with the symbol ' $\divideontimes$ '.

  These are the *candidates* for inflection points.
- Find the intervals where $f''(x) > 0$ (f is concave up) and $f''(x) < 0$ (f is concave down), using the now-familiar procedure:

  Draw a number line labeled *Sign of* $f''(x)$. On it, indicate all the places where f'' is *not continuous*, and all the places where $f''(x) = 0$.

  Choose a test point T in each interval, and see if $f''(T)$ is positive or negative.

  Use this information to sketch the graph.
- Fill in any necessary details, such as x-axis intercept(s), y-axis intercept, asymptote information, and behavior at $\pm\infty$.

global maximum; global minimum

Thus far in this text, we have discussed:

- maximum and minimum values *on an interval*
- *local* maximum and minimum values

Sometimes, it is of interest to know if a function attains a maximum or minimum value, as the inputs are allowed to vary over the *entire domain* of f. If such an extreme value exists, it is called a *global extreme value.*

The precise definition follows.

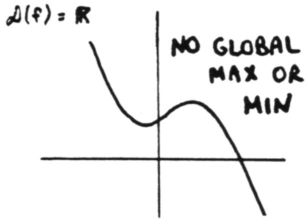
DEFINITION

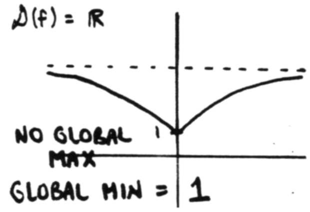
global maximum; global minimum

Let f be a function with domain $\mathcal{D}(f)$.

If there exists $c_1 \in \mathcal{D}(f)$ such that $f(c_1) \le f(x) \ \ \forall\, x \in \mathcal{D}(f)$, then the number $f(c_1)$ is the *global minimum* for f.

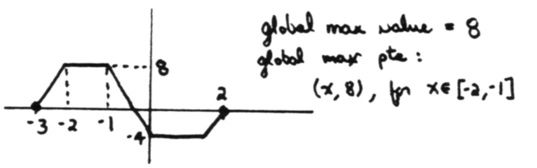
If there exists $c_2 \in \mathcal{D}(f)$ such that $f(c_2) \ge f(x) \ \ \forall\, x \in \mathcal{D}(f)$, then the number $f(c_2)$ is the *global maximum* for f.

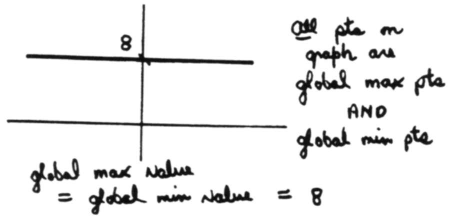




values (numbers) versus points

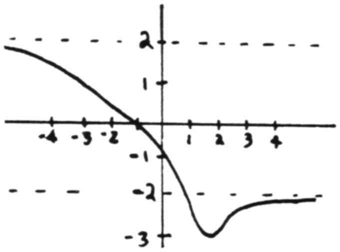
Note that if a global minimum or maximum *value* exists, then it must be unique. However, this value may be taken on by *more than one input*, as the examples below illustrate. As usual, one is often interested in knowing the input(s) that give rise to global extreme values. Thus, one frequently speaks of, say, a *global maximum point.*

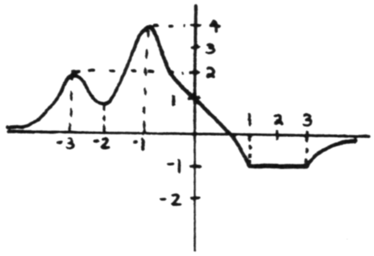




EXERCISE 1

♣ Decide if the graphs shown below have a global maximum value; global minimum value. If so, list all global maximum point(s); all global minimum point(s). Assume that the domain of each function is $\mathbb{R}$.





EXERCISE 2

If the following sentences are false, make a sketch which illustrates how they can fail.

♣ 1. True or False: If $(c, f(c))$ is a local maximum point for f, then it is a global maximum point for f.

♣ 2. True or False: If $(c, f(c))$ is a global maximum point for f, then it is a local maximum point for f.

♣ 3. True or False: If the number M is a global maximum value for f, then it is unique.

♣ 4. True or False: If the point (c, M) is a global maximum point for f, then it is unique.

Now, lots of graphing examples!!

EXAMPLE

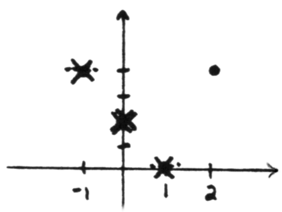
graphing a polynomial

Problem: Completely graph:

$$P(x) = x^3 - 3x + 2$$

- $\mathcal{D}(f) = \mathbb{R}$. Plot a few simple points:

x	P(x)
0	2
1	0
-1	4
2	4



- Find the first derivative:

$$P'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$$

Observe that $\mathcal{D}(f') = \mathbb{R}$, so the only critical points come from places where $f'(x) = 0$. Be sure to write down complete mathematical sentences.

$$\begin{aligned} P'(x) = 0 \quad &\iff \quad 3(x-1)(x+1) = 0 \\ &\iff \quad x = 1 \ \text{ or } \ x = -1 \end{aligned}$$

So, $(1, f(1)) = (1, 0)$ and $(-1, f(-1)) = (-1, 4)$ are critical points. Plot these with an $\times$ to emphasize that they correspond to places where there is a horizontal tangent line.

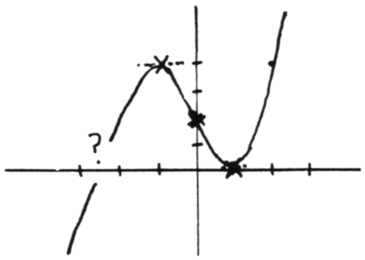
- Find the second derivative:

$$P''(x) = 6x$$

Again, $\mathcal{D}(f'') = \mathbb{R}$, so the only candidates for inflection points occur when $f''(x) = 0$.

$$\begin{aligned} P''(x) = 0 \quad &\iff \quad 6x = 0 \\ &\iff \quad x = 0 \end{aligned}$$

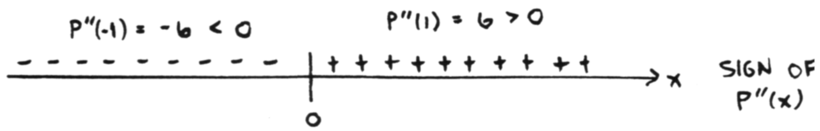
Thus, $(0, f(0)) = (0, 2)$ is the only candidate for an inflection point. Plot this point with an $⋇$ to emphasize that there may be an inflection point here.

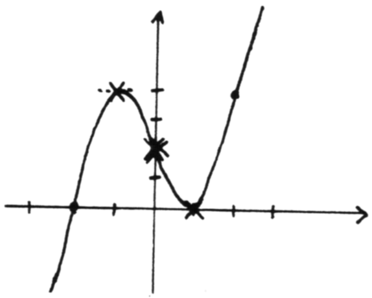


- Investigate the sign of the second derivative:

  P'' is continuous everywhere, and is zero only at $x = 0$. Make a number line, indicating the point $x = 0$. This yields two subintervals, $(-\infty, 0)$ and $(0, \infty)$.

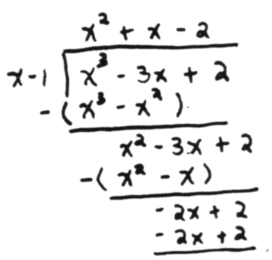
  Choose a 'test point' from each of these intervals.





  Thus, P'' is positive on $(0, \infty)$ and negative on $(-\infty, 0)$, so P is concave up on $(0, \infty)$ and concave down on $(0, -\infty)$. Use this information to fill in the majority of the graph.

- Fill in any missing details. Here, it would be nice to know the second x-axis intercept. We can always 'zero in' on it, using the Intermediate Value Theorem. However, in this case, we can do even better. Since $x = 1$ is a root of P, $x - 1$ must be a factor of P. Do a long division:



  Thus, $P(x) = (x-1)(x^2 + x - 2) = (x-1)^2(x+2)$. The remaining x-axis intercept occurs at $x = -2$.

- Once the graph of P is complete, read off all this important information:

  $(-1, 4)$ is a local maximum

  $(1, 0)$ is a local minimum

  no global maximum, no global minimum

  $(0, 2)$ is an inflection point

  concave down on $(-\infty, 0)$

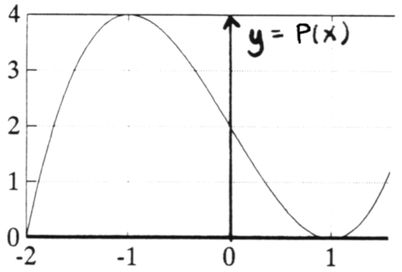
  concave up on $(0, \infty)$

  increasing on $(-\infty, -1) \cup (1, \infty)$

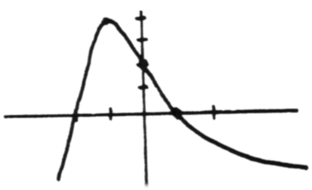
  decreasing on $(-1, 1)$

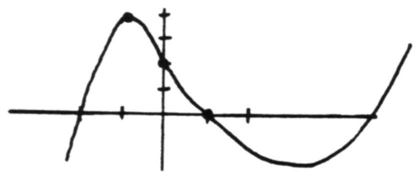
  Note that it was *not* necessary to investigate the sign of P' to find out where P increases and decreases.

  A graph of P is shown.



EXERCISE 3 Reconsider the previous example. It was found that P is concave up on $(0, \infty)$. Why couldn't the graph look like the two situations shown below? Comment.





checking behavior at infinity

In the last step of the previous example, one final check could have been made: the graph shows that as $x \to \infty$, $f(x) \to \infty$; and as $x \to -\infty$, $f(x) \to -\infty$. It would be prudent to verify that the function f really behaves this way.

approximating polynomials by their highest order term

To this end, an important property of polynomials is needed. Informally:

When x is large (positive or negative), then a polynomial P is well approximated by its highest order term.

Another way to state this is:

For large x, the highest order term of a polynomial dominates.

The *highest order term* of a polynomial in x is the term with the greatest exponent on x.

$x \gg 0$ means x is large and positive

$x \ll 0$ means x is large and negative

The phrase 'x is large' is sometimes used to mean that x is a number that is very, very far from zero on the number line. That is, either x is positive and $|x|$ is much greater than zero; or x is negative and $|x|$ is much greater than zero. Thus, one might say that both 10^7 and -2^{36} are 'large' numbers.

If x is large, then $P(x) = x^3 - 3x + 2$ is well approximated by the simpler polynomial $\tilde{P}(x) = x^3$. That is, for large x:

$$x^3 - 3x + 2 \approx x^3$$

(The symbol '$\approx$' is read as *is approximately equal to*.)

When x is large and positive, so is x^3. Thus, so must be $x^3 - 3x + 2$.

When x is large and negative, so is x^3. Thus, so must be $x^3 - 3x + 2$.

The sentence '$x \gg 0$' is read as 'x is much greater than zero'. So instead of saying 'x is large and positive', one can equivalently say '$x \gg 0$'.

The sentence '$x \ll 0$' is read as 'x is much less than zero'. So instead of saying 'x is large and negative', one can equivalently say '$x \ll 0$'.

more precisely: investigate a limit!

This idea of 'approximation by the highest order term' can be made precise. Consider an arbitrary polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \ , \qquad (*)$$

where $a_n \neq 0$. The highest order term is $a_n x^n$.

It is possible to get $P(x)$ *as close to $a_n x^n$ as desired*, by making x sufficiently large. To see that this is true, divide both sides of (*) by x^n, obtaining:

$$\frac{P(x)}{x^n} = a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}$$

For large (enough) values of x, all the terms on the right-hand side, except a_n, will be close to zero. That is, as x approaches $+\infty$ or $-\infty$, $\frac{P(x)}{x^n}$ approaches a_n. And when $\frac{P(x)}{x^n}$ is close to a_n, then $P(x)$ is close to $a_n x^n$. This idea will be made yet more precise in the final section of this chapter.

EXAMPLE

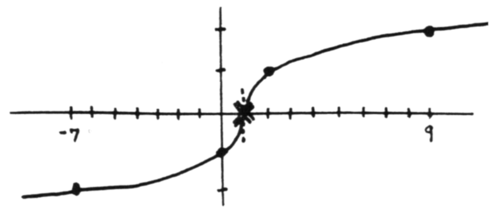
Problem: Completely graph:

$$f(t) = (t-1)^{1/3}$$

- $\mathcal{D}(f) = \mathbb{R}$. Plot a few points:

t	$f(t)$
0	-1
1	0
2	1
9	2

t	$f(t)$
-7	-2



- $f'(t) = \frac{1}{3}(t-1)^{-2/3}$

  $\mathcal{D}(f') = \{x \,|\, x \neq 1\}$. Observe that f *is* defined at 1, but f' is *not* defined at 1. Thus, $(1, f(1)) = (1, 0)$ is a critical point. As x approaches 1 (from either side), $f'(x) \to \infty$, so there is a *vertical tangent line* at the point $(1, 0)$. Indicate this on the graph using the symbol $\times$.

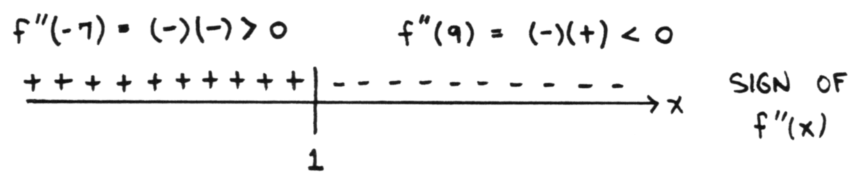
  f' is never equal to 0, so there are no other critical points.

- $f''(t) = -\frac{2}{9}(t-1)^{-5/3}$

  Again, $\mathcal{D}(f'') = \{x \,|\, x \neq 1\}$. So f'' is *not* defined at $x = 1$, but f is. Thus, $(1, 0)$ is also a candidate for an inflection point. Put a ⋇ over this point, to remind us of this fact.

  f'' is never equal to 0, so there are no other candidates for inflection points.

- f'' is continuous everywhere except at 1, and is never 0. Thus, one need only check the sign of f'' (plus or minus) on the intervals below.



- Details: check behavior at infinity.

  For large values of t:

  $$(t-1)^{1/3} \approx t^{1/3} = \sqrt[3]{t}$$

  So, as $t \to \infty$, $f(t) \to \infty$. And, as $t \to -\infty$, $f(t) \to -\infty$.

- Read off all important information:

  no local maxima or minima

  no global maximum or minimum

  $(1, 0)$ is an inflection point

  x-axis intercept: $(1, 0)$

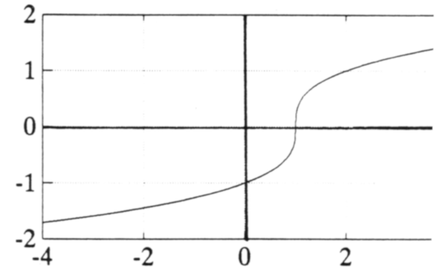
  y-axis intercept: $(0, -1)$

  concave up on $(-\infty, 1)$

  concave down on $(1, \infty)$

  increasing on $(-\infty, \infty)$

  A graph of f is shown at right.



EXAMPLE

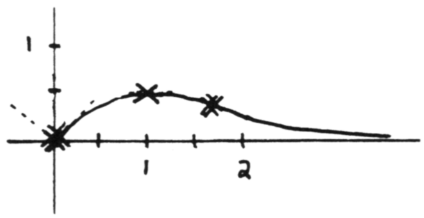
Problem: Completely graph:

$$g(x) = \frac{|x|}{x^2+1}$$

- $\mathcal{D}(g) = \mathbb{R}$, since x^2+1 is never zero. Note that g is an even function, since:

$$g(-x) = \frac{|-x|}{(-x)^2+1} = \frac{|x|}{x^2+1} = g(x)$$

  Thus, g only needs to be graphed on $(0,\infty)$; the rest is filled in from symmetry.
- For $x > 0$, $|x| = x$, so that $g(x) = \frac{x}{x^2+1}$ and:



$$\begin{aligned} g'(x) &= \frac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2} \\ &= \frac{1-x^2}{(x^2+1)^2} \\ &= \frac{(1-x)(1+x)}{(x^2+1)^2} \end{aligned}$$

Remember that this formula only holds for $x > 0$. When $x = 1$, there is a horizontal tangent line. So, $(1, g(1)) = (1, \frac{1}{2})$ is a critical point.

Is there a tangent line at $x = 0$? Note that:

$$\lim_{x\to 0^+} g'(x) = \lim_{x\to 0^+} \frac{(1-x)(1+x)}{(x^2+1)^2} = \frac{(1)(1)}{(1)^2} = 1$$

So, as x approaches zero from the right, the tangent lines have slopes that approach 1. Sketch in a dashed line with slope 1 to the right of zero, as shown.

By symmetry, as zero is approached from the left, the tangent lines have slopes that approach -1. Thus, there is a 'kink' at zero. That is, $g'(0)$ does not exist. So, $(0, g(0)) = (0,0)$ is also a critical point.

Observe that it has been shown that g' is not continuous at 0. Indeed, g' has a nonremovable discontinuity at $x = 0$.

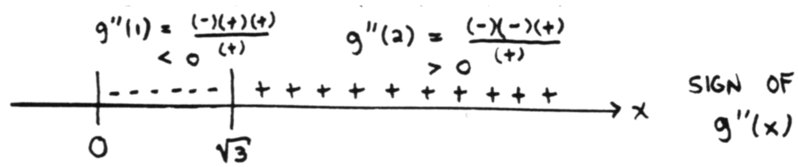
- Since g' is not continuous at 0, g' is not differentiable at 0. (♣ Why?) That is, $g''(0)$ does not exist, and is a candidate for an inflection point.

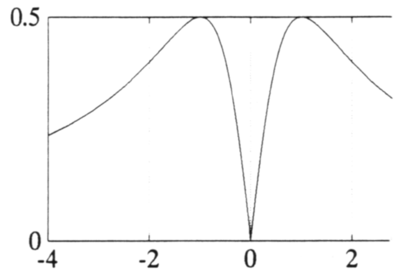
  For $x > 0$:

$$\begin{aligned} g''(x) &= \frac{(x^2+1)^2(-2x) - (1-x^2)2(x^2+1)(2x)}{(x^2+1)^4} \\ &= \frac{-2x(x^2+1)[(x^2+1) + 2(1-x^2)]}{(x^2+1)^4} \\ &= \frac{-2x(3-x^2)}{(x^2+1)^3} \\ &= \frac{-2x(\sqrt{3}-x)(\sqrt{3}+x)}{(x^2+1)^3} \end{aligned}$$

  When $x = \sqrt{3} \approx 1.7$, $g''(x)$ is zero. Thus, $(1.7, g(1.7)) = (1.7, 0.4)$ is an (approximate) candidate for an inflection point.

- Investigate the sign of g'' on $(0,\sqrt{3})$ and $(\sqrt{3},\infty)$:





Thus, g is concave down on $(0,\sqrt{3})$ and concave up on $(\sqrt{3},\infty)$.

- Details: check behavior at infinity.

For $x \gg 0$:

$$\frac{x}{x^2+1} \approx \frac{x}{x^2} = \frac{1}{x}$$

Thus, as $x \to \infty$, $f(x) \to 0$.

- Read off all important information:

$(0,0)$ is a local and global minimum
$(1,0.5)$ and $(-1,0.5)$ are local and global maxima
$(\sqrt{3},\frac{\sqrt{3}}{4})$ and $(-\sqrt{3},\frac{\sqrt{3}}{4})$ are inflection points
concave up on $(-\infty,-\sqrt{3}) \cup (\sqrt{3},\infty)$
concave down on $(-\sqrt{3},0) \cup (0,\sqrt{3})$
increasing on $(-\infty,-1) \cup (0,1)$
decreasing on $(-1,0) \cup (1,\infty)$
A graph of g is shown at left.

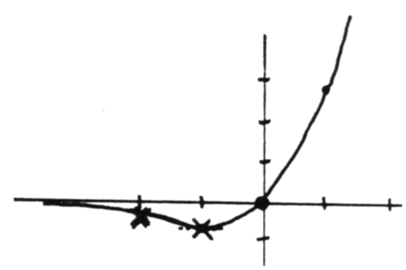
EXAMPLE

Problem: Completely graph:

$$f(x) = xe^x$$

- $\mathcal{D}(f) = \mathbb{R}$. Plot a few points:

x	f(x)
0	0
1	e ≈ 2.7
-1	-1/e ≈ -.4



- $f'(x) = xe^x + (1)e^x = e^x(x+1)$

$\mathcal{D}(f') = \mathbb{R}$

$$\begin{aligned} f'(x) = 0 \quad &\iff \quad e^x(x+1) = 0 \\ &\iff \quad x+1 = 0 \\ &\iff x = -1 \end{aligned}$$

Thus, $(-1, f(-1)) = (-1,-e^{-1}) = (-1,-\frac{1}{e}) \approx (-1,-0.4)$ is the only critical point.

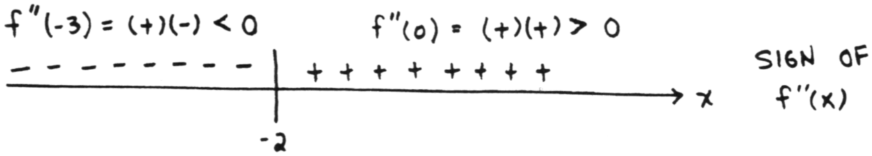
- $f''(x) = e^x(1) + e^x(x+1) = e^x(x+2)$

$\mathcal{D}(f'') = \mathbb{R}$

$$f''(x) = 0 \quad \iff \quad x = -2\ ,$$

so $(-2,-2e^{-2}) \approx (-2,-0.3)$ is the only candidate for an inflection point.

- Investigate the sign of the second derivative:



- Details: Note that

$$f(x) = 0 \iff xe^x = 0 \iff x = 0\ ,$$

so the only x-axis intercept is at 0.

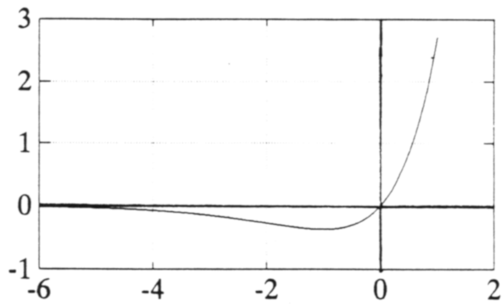
As $x \to \infty$, $f(x) \to \infty$.

As $x \to -\infty$, we run into a '$(-\infty)(0)$' situation, which requires further investigation. In this case, plotting some additional points, and using the fact that f cannot cross the x-axis again, we conclude that as $x \to -\infty$, $f(x) \to 0$.

- $(-1, -\frac{1}{e})$ is a local and global minimum

  $(-2, -\frac{2}{e^2})$ is an inflection point
  concave up on $(-2, \infty)$
  concave down on $(-\infty, -2)$
  increasing on $(-1, \infty)$
  decreasing on $(-\infty, -1)$

  A graph of f is shown below.



EXERCISE 4

Completely graph the following functions:

♣ 1. $P(x) = 3x^4 + 4x^3 - 12x^2 + 1$

♣ 2. $f(t) = (t + 2)^{1/5}$

♣ 3. $g(x) = \frac{|x|}{x^2 - 1}$

♣ 4. $f(x) = xe^{-x}$

Read off all this information from your graphs:

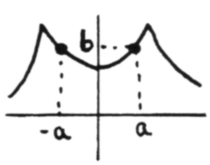
local maxima and minima
global maxima and minima
inflection points
x and y-axis intercepts (approximate, if necessary)
open intervals on which the graph is concave up and down
open intervals on which the graph is increasing and decreasing

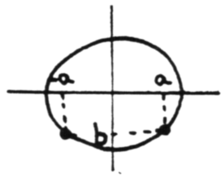
ALGEBRA REVIEW

even and odd functions

DEFINITION
symmetry about the y-axis

If a graph has the property that whenever (a, b) is on the graph, so is $(-a, b)$, then the graph is *symmetric about the y-axis.*





Observe that if such a graph is folded the graph along the y-axis, the part of the graph to the right of the y-axis coincides with the part to the left. Why is this? Answer: By folding along the y-axis, one is *identifying* points that have the same magnitude x-values. For example, after folding, $x = 2$ ends up on top of $x = -2$. And, $x = 5$ ends up on top of $x = -5$. For a graph that is symmetric about the y-axis, such points have exactly the same y-values, so the points coincide.

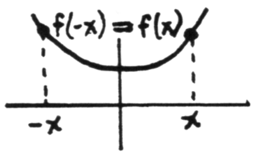
There is an equivalent characterization of symmetry about the y-axis, if one happens to be working with a function:

DEFINITION
even functions

If a function f satisfies the property that

$$f(-x) = f(x) \quad \forall\ x \in \mathcal{D}(f) ,$$

then f is an *even* function, and its graph is symmetric about the y-axis.



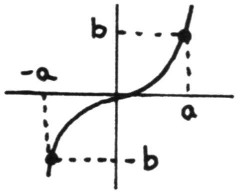
For example, $f(x) = x^4$ is an even function. To see this, one need only verify that:

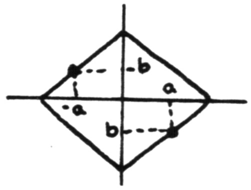
$$f(-x) = (-x)^4 = x^4 = f(x)$$

♣ Is $f(x) = x^6 + 2x^2$ an even function? How about $g(x) = \frac{1}{x^2+1}$?

DEFINITION
symmetry about the origin

If a graph satisfies the property that whenever (a, b) is on the graph, so is $(-a, -b)$, then the graph is *symmetric about the origin.*





Note that if such a graph is folded *twice*—once along the x-axis, and once along the y-axis—then the parts of the graph coincide.

♣ Think about why this 'coinciding' takes place.

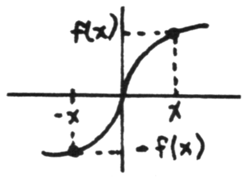
There is an equivalent characterization of symmetry about the origin, if one happens to be working with a function:

DEFINITION
odd functions

If a function f satisfies the property that

$$f(-x) = -f(x) \ \ \forall\, x \in \mathcal{D}(f) \ ,$$

then f is an *odd* function, and its graph is symmetric about the origin.



So if f is an odd function, then whenever $(x, f(x))$ is on the graph, so is $(-x, -f(x))$.

♣ Show that $f(x) = x^3$ is an odd function; graph it.

♣ Is $f(x) = \dfrac{1}{x^3 - x}$ an odd function? How about $g(x) = \dfrac{x}{x^3 - x}$?

QUICK QUIZ
sample questions

1. Sketch the graph of a function that has a global maximum value of 10; there should be 3 global maximum points.
2. When $x \gg 0$, what does the graph of $P(x) = 127 - 3x + x^4 - 6x^7$ look like? How about when $x \ll 0$?
3. Is $f(x) = x^5 - x$ an even function? An odd function? Be sure to support your answers.
4. Completely graph $f(x) = 6x^2 - 7x - 3$, using the systematic approach discussed in this section.

KEYWORDS
for this section

A systematic approach to graphing a function, symmetry about the y-axis, even functions, symmetry about the origin, odd functions, global maximum and minimum values, global maximum and minimum points, checking behavior at infinity, approximating polynomials by their highest order term.

END-OF-SECTION EXERCISES

♣ Re-do each of the graphing examples from this section, *without looking at the text.* If you get stuck, then study the example, and try it on your own again.