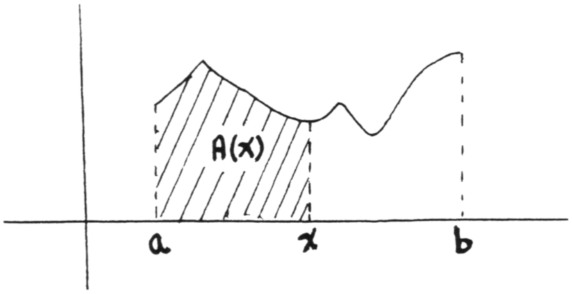
7.1 Using Antiderivatives to find Area

Introduction

In this section a formula is obtained for finding the area of the region bounded between the graph of a *continuous, nonnegative* function f and the x-axis. As mentioned in the previous chapter, it is seen that the antiderivatives of f play a *crucial* role in this process.

finding the area under the graph of a nonnegative, continuous function f

Let f be a function that is continuous on $[a, b]$. Also suppose that f is nonnegative, so that its graph lies on or above the x-axis. In this case, it makes sense to talk about the area under the graph of f; we seek the area between $x = a$ and $x = b$.



the area function; A(x)

First, define:

$$A(x) := \text{the area under the graph of } f, \text{ from } a \text{ to } x$$

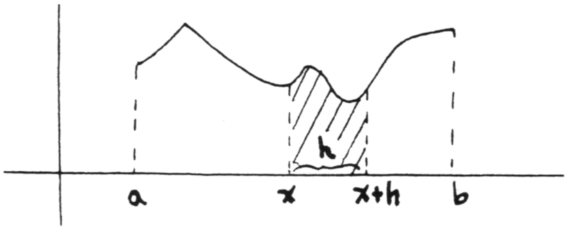
Observe that $A(a) = 0$, and $A(b)$ is the area being sought.

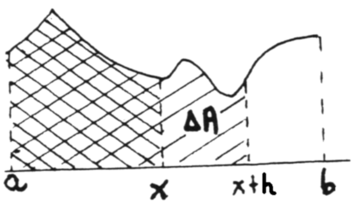
in the pictures, h is positive

Now, let x be a number between a and b, and let h be a small *positive* number. In the exercises accompanying this section, you will consider the case where h is a small *negative* number.

ΔA; a little piece of area

Focus attention on the little piece of area between x and $x+h$, as shown below.





This area can be obtained as follows: take the area under the graph from a to $x + h$, and subtract off the area from a to x. What's left is the area under the graph between x and $x + h$, as shown.

Thus, this little piece of area can be written in terms of the area function A as:

$$\Delta A := A(x + h) - A(x)$$

The symbol ΔA is read as '*delta A*' and denotes a *change in A*.

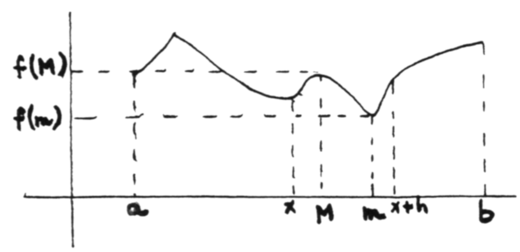
EXERCISE 1

♣ 1. If h is a small negative number, where is $x + h$ in relation to x?

♣ 2. Make a sketch showing x and $x + h$. What is the correct formula for ΔA in this case?

using the Max-Min Theorem

By hypothesis, f is continuous on the entire interval $[a, b]$, so it is also continuous on the subinterval $[x, x+h]$. Therefore, the Max-Min Theorem guarantees that f attains a minimum value $f(m)$ and a maximum value $f(M)$ on $[x, x+h]$, as illustrated below. Observe that both m and M come from the interval $[x, x+h]$.



under- and over-approximating the area with rectangles

The actual area ΔA of the little piece under inspection is under-approximated by the rectangle of height $f(m)$ and width h. Also, ΔA is over-approximated by the rectangle of height $f(M)$ and width h. That is:

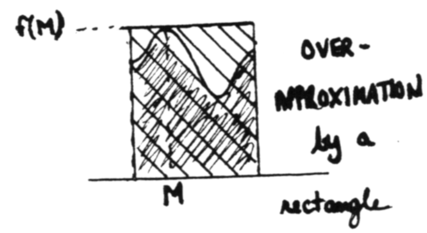
$$f(m) \cdot h \le \Delta A \le f(M) \cdot h$$

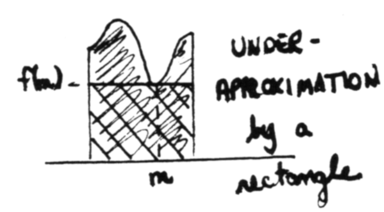
Division by the positive number h yields

$$f(m) \le \frac{\Delta A}{h} \le f(M) \ ,$$

and substituting in the definition of ΔA yields:

$$f(m) \le \frac{A(x+h) - A(x)}{h} \le f(M)$$





Be aware!

The numbers m and M depend on: the function f the number x the number h

What is about to be said applies to *both* m and M. For simplicity, it is stated only for m.

It's important that you understand that the number m depends on:

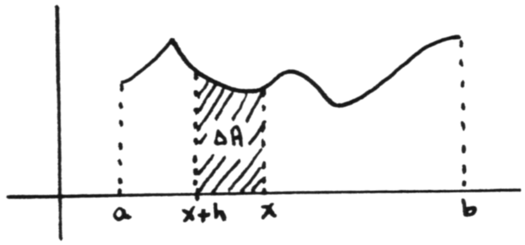
- the function (f) that you're working with
- the small interval $[x, x+h]$ that is currently under investigation; this interval depends on both x and h

Change any of these (f , x , or h) and the number m could change!

For this reason, a name like '$m_{f,x,h}$' (with three subscripts) might be better than just m. But then the notation would be so cumbersome that it could make things appear harder than they really are! So, we'll stick with just 'm'.

EXERCISE 2

You should have discovered in the previous exercise that if $h < 0$, then $\Delta A = A(x) - A(x+h)$, and the picture becomes the one shown below:



♣ 1. Why is the area of the under-approximating rectangle given by the formula $f(m) \cdot (-h)$ in this case?

♣ 2. What is the formula for the area of the over-approximating rectangle?

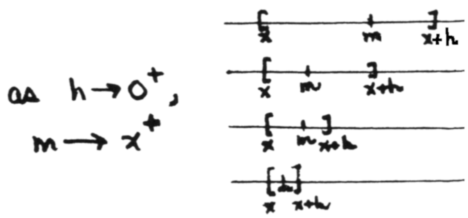
♣ 3. Provide a justification for each step in the mathematical sentence below. Remember that $h < 0$, and $\Delta A = A(x) - A(x+h)$.

$$\begin{aligned} f(m)(-h) \le \Delta A \le f(M)(-h) \quad &\iff \quad f(m) \le \frac{\Delta A}{-h} \le f(M) \\ &\iff \quad f(m) \le \frac{A(x) - A(x+h)}{-h} \le f(M) \\ &\iff \quad f(m) \le \frac{A(x+h) - A(x)}{h} \le f(M) \end{aligned}$$

Thus, precisely the same inequality is obtained as when h is positive.

let h approach 0; then m must approach x

Now let h approach 0 (from the right-hand side, since h is positive). Remember that m is trapped in the interval $[x, x+h]$, so as h approaches zero, m is forced to get close to x. That is, as $h \to 0^+$, it must be that $m \to x^+$.



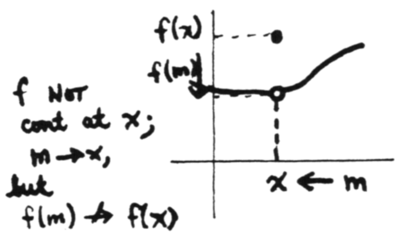
Note: Here, we're holding x fixed and letting h change. Since h is changing, m can change! The same label, 'h', is used in all four sketches above, even though h is getting smaller. The same label, 'm', is used, even though it can change. This can be confusing—same labels, different numbers—so be aware!

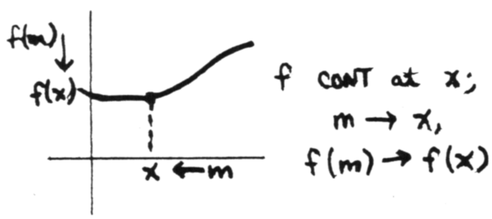
EXERCISE 3

♣ Rewrite the previous paragraph so that it applies when $h < 0$.

using the continuity of f

By hypothesis, f is continuous at x. Therefore, when the inputs are close to x, the corresponding outputs must be close to $f(x)$. In particular, when m is close to x, $f(m)$ must be close to $f(x)$. More precisely, as $m \to x^+$, we must have $f(m) \to f(x)$.





as h approaches 0, both m and M must get close to x

Similarly, since M is trapped between x and $x+h$, as h approaches 0, M must approach x. And as M gets close to x, the continuity of f at x tells us that $f(M)$ approaches $f(x)$.

the quotient $\frac{A(x+h)-A(x)}{h}$ is pinched between numbers that are both going to $f(x)$

Reconsider the previous inequality in light of our new information:

$$\overset{\nearrow f(x)}{f(m)} \le \frac{A(x+h)-A(x)}{h} \le \overset{\nearrow f(x)}{f(M)}$$

As h approaches 0 (from the right-hand side), both $f(m)$ and $f(M)$ are approaching $f(x)$. So the quotient

$$\frac{A(x+h)-A(x)}{h}$$

is pinched between numbers which are *both* going to the *same number*, $f(x)$! Therefore, $\frac{A(x+h)-A(x)}{h}$ must also be getting close to $f(x)$! (This observation is sometimes formalized in a result called the *Pinching Theorem for Limits*.) That is, it must be that:

$$\lim_{h\to 0^+} \frac{A(x+h)-A(x)}{h} = f(x)$$

EXERCISE 4

♣ Rewrite the necessary paragraphs, and conclude that:

$$\lim_{h\to 0^-} \frac{A(x+h)-A(x)}{h} = f(x)$$

the limit $\displaystyle\lim_{h\to 0} \frac{A(x+h)-A(x)}{h}$ exists

Now it is known that

$$\lim_{h\to 0^+} \frac{A(x+h)-A(x)}{h} = f(x)\ ;$$

and, if you've been doing the exercises, it is also known that:

$$\lim_{h\to 0^-} \frac{A(x+h)-A(x)}{h} = f(x)$$

Putting these two pieces of information together, we conclude that the two-sided limit exists and equals $f(x)$:

$$\lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = f(x)$$

But when the limit

$$\lim_{h\to 0} \frac{A(x+h)-A(x)}{h}$$

exists, it is given a special name: $A'(x)$! So it is now known that:

$$A'(x) = f(x)$$

That is, the area function A is a function which, when differentiated, yields f. That is, A is an antiderivative of f.

$A(x)$ is an antiderivative of $f(x)$

The fact just discovered is so important that it is worth repeating. *The area function A is an antiderivative of f.* In particular, it has been shown that whenever f is continuous and nonnegative on $[a, b]$, *an antiderivative of f always exists*! This is an extremely beautiful and important result.

now we know what **all** *the antiderivatives look like*

Getting our hands on *one* antiderivative is always the hard part; now we know what *all* the antiderivatives of f must look like—they must differ from A by at most a constant. That is, if F denotes *any* antiderivative of A, then:

$$A(x) = F(x) + C \qquad (*)$$

solving for the constant C

Remember that we want to find $A(b)$, since this represents the area under the graph of f between a and b. Using the fact that $A(a) = 0$, equation (*) yields

$$0 = A(a) = F(a) + C$$

so that $C = -F(a)$. Then (*) can be rewritten as:

$$A(x) = F(x) - F(a)$$

Now, letting x equal b, we obtain:

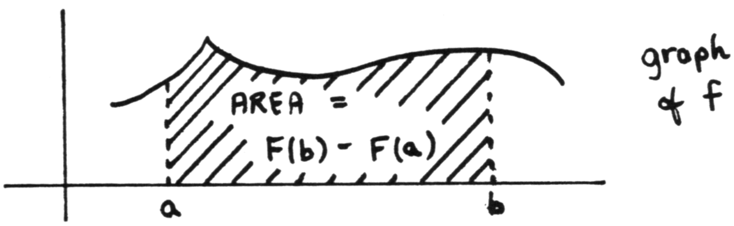
$$\text{desired area} = A(b) = F(b) - F(a)$$

This is the formula for the desired area, given in terms of *any* antiderivative of f. The result is summarized below.

formula for the area beneath the graph of a nonnegative, continuous function f on $[a, b]$

Let f be nonnegative and continuous on the interval $[a, b]$. Let F be *any* antiderivative of f on $[a, b]$. Then:

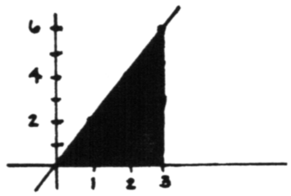
$$\text{the area under the graph of } f \text{ on } [a,b] \ = \ F(b) - F(a)$$



EXAMPLE

testing the formula in a case where the answer is already known

It's always a good idea to test a new result in a situation where you can find the answer by alternate means. So let's find the area under the graph of $f(x) = 2x$ between $x = 0$ and $x = 3$.



Calculus is certainly not needed, since the area is just a triangle:

$$\frac{1}{2}(\text{base})(\text{altitude}) = \frac{1}{2}(3)(6) = 9$$

Now, use the formula. An antiderivative of $f(x) = 2x$ is needed; the easiest one is $F(x) = x^2$. Then,

$$F(b) - F(a) = F(3) - F(0) = 3^2 - 0 = 9\ ,$$

which agrees with the first result.

EXERCISE 5

♣ 1. Show that $F(x) = x^2 + 7$ is an antiderivative of $f(x) = 2x$.

♣ 2. Find the area discussed in the previous example, using the antiderivative $F(x) = x^2 + 7$. What happens to the '7'?

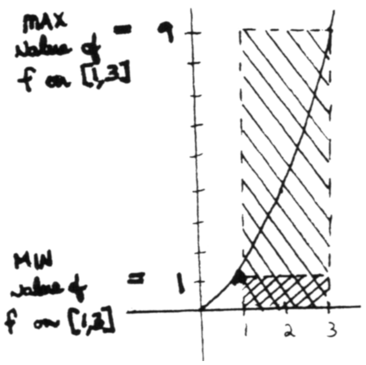
EXERCISE 6

Find the area under the graph of $f(x) = 2x$ between $x = 1$ and $x = 4$ in two ways:

♣ 1. Show that the desired area is a trapezoid; find the area of this trapezoid.

♣ 2. Use an antiderivative of f to find the area.

EXAMPLE



Problem: Find the area beneath the graph of $f(x) = x^2$ on $[1, 3]$.

Solution: Here, the area of the region is *not* easily obtainable from geometry. However, we can get some rough bounds on the desired area, as follows.

The minimum value of f on $[1, 3]$ is $1^2 = 1$. Thus, the desired area is under-approximated by a rectangle of width $3 - 1 = 2$ and height 1.

The maximum value of f on $[1, 3]$ is $3^2 = 9$. Thus, the desired area is over-approximated by a rectangle of width 2 and height 9. Together:

$$(1)(2) \le \text{actual area} \le (9)(2)$$

The actual area must lie between 2 and 18. Also, from the sketch, we expect the actual area to be near the middle of this range of numbers.

applying the formula

Now apply the formula. We need any antiderivative of $f(x) = x^2$; take $F(x) = \frac{x^3}{3}$, since it's the simplest one. Then:

$$F(b) - F(a) = F(3) - F(1) = \frac{3^3}{3} - \frac{1^3}{3} = 9 - \frac{1}{3} = 8\frac{2}{3}$$

The answer is certainly believable, based on the earlier estimates.

EXERCISE 7

♣ 1. Consider the function $f(x) = x^2$ on the interval $[2, 5]$. As in the previous example, get an under-approximation and an over-approximation of the area under f on $[2, 5]$.

♣ 2. Find the area, using an antiderivative of f.

♣ 3. Find the area, using a different antiderivative of f.

EXERCISE 8

♣ Use calculus to find the area under the graph of $f(x) = x^2$ on $[-2, -1]$. Here, $[a, b] = [-2, -1]$, so $a = -2$ and $b = -1$. Make a sketch of the graph of f, and the area that you are finding.

EXERCISE 9

♣ 1. Graph $f(x) = -x^2$. Show the area trapped between the graph of f and the x-axis on $[1, 3]$.

♣ 2. Using any antiderivative F of f, compute $F(3) - F(1)$. How does your answer compare to the area under the graph of $f(x) = x^2$ on $[1, 3]$?

♣ 3. Make a conjecture, based on this example.

QUICK QUIZ
sample questions

1. Suppose $h > 0$, and f is continuous on the interval $[x, x + h]$. What does the Max-Min Theorem guarantee?
2. Under what condition(s) does a function f have the property that as $x \to a$, $f(x) \to f(a)$?
3. Make a sketch that illustrates a function f, and $a \in \mathcal{D}(f)$, for which $f(x) \not\to f(a)$ as $x \to a$.
4. Find the area under the graph of $y = 3x^2$ on the interval $[0, 2]$.
5. Suppose f is continuous and nonnegative on $[c, d]$, and F is an antiderivative of f. Give a formula for the area under the graph of f on $[c, d]$.

KEYWORDS
for this section

Finding the area under the graph of a continuous, nonnegative function f on the interval $[a, b]$; a formula for this area in terms of any antiderivative F of f.

END-OF-SECTION EXERCISES

In each problem below, an area is described.

♣ Sketch the area that is described.

♣ Approximate the area in any reasonable way.

♣ Use calculus to find the area.

1. area bounded between the graph of $y = \ln x$ and the x-axis on the interval $[1, e]$
2. area under the graph of $y = \frac{1}{t}$ on $[1, 2]$
3. area bounded by the graph of $y = \sqrt{x}$, the x-axis, the line $x = 1$, and the line $x = 4$
4. area bounded by the graph of $y = x^2 + 1$, the line $y = 1$, the y-axis, and the line $x = 1$