7.2 The Definite Integral

the definite integral

In the previous section, it was found that if a function f is continuous and nonnegative, then the area under the graph of f on $[a, b]$ is given by $F(b) - F(a)$, where F is any antiderivative of f.

This result is usually expressed in terms of an integral, called *the definite integral of f on $[a, b]$*, and denoted by:

$$\int_a^b f(x)\,dx$$

(Read $\int_a^b f(x)\,dx$ as '*the definite integral of f, from a to b*'.)

In this section, study of the definite integral begins.

the actual definition of $\int_a^b f(x)\,dx$ is a bit complicated

The actual *definition* of the definite integral $\int_a^b f(x)\,dx$ is a bit complicated, due mainly to the fact that (not surprisingly!) it is defined in terms of a *limit*. The precise definition of $\int_a^b f(x)\,dx$ is presented in the next section. This definition reveals the following facts (which you can take on faith for the moment, and start understanding *now*):

For a continuous function f,

- $\int_a^b f(x)\,dx$ is a NUMBER; and
- if f happens to be *nonnegative* on $[a, b]$, then this number $\int_a^b f(x)\,dx$ has a very nice interpretation; it gives the area under the graph of f on $[a, b]$. Since $F(b) - F(a)$ gives this same area, where F is any antiderivative of f, we can in this case write:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

lower limit of integration;

upper limit of integration

The definite integral

$$\int_a^b f(x)\,dx$$

and the indefinite integral

$$\int f(x)\,dx$$

have similar appearances. The only difference is that the definite integral has numbers a and b adorning the integral sign. These two new components have names:

- a is called the *lower limit of integration*
- b is called the *upper limit of integration*

comparing the definite and indefinite integrals

Since the appearance of the two integrals is so similar, you should be asking yourself the following questions: Why the similar appearance? How are these integrals the same? How are they different? Here are some answers to these questions.

How are the integrals different?

The definite integral $\int_a^b f(x)\,dx$ is a NUMBER. If f is nonnegative on $[a,b]$, then this number has a nice interpretation as the area under the graph of f on $[a,b]$. However, the indefinite integral $\int f(x)\,dx$ is an INFINITE CLASS OF FUNCTIONS; all the antiderivatives of f. So, in one sense, the two integrals are very, very different.

How are the integrals the same?

In another sense, however, they are very much the same. It will be seen that if JUST ONE antiderivative of f is known, then the definite integral can be computed. This fact has already been established in a special case—when f is continuous and nonnegative on $[a,b]$—and we will see that it actually holds for *any* continuous function f. This is precisely the content of the Fundamental Theorem of Integral Calculus (to be presented momentarily); and is the justification for the similarity in the appearance of the two integrals.

EXERCISE 1

TRUE or FALSE:

♣ 1. The definite integral $\int_a^b f(x)\,dx$ is a function.

♣ 2. The number $\int_1^5 x^2\,dx$ gives the area under the graph of x^2 on the interval $[1,5]$.

♣ 3. For a continuous function f, $\int_a^b f(x)\,dx = \int f(x)\,dx$.

♣ 4. This text has not yet presented the actual definition of $\int_a^b f(x)\,dx$.

We've run across a situation before where a (precise) definition was hard to work with, and—fortunately—we could often get away with NOT working with the definition. Remember the *definition* of the derivative of a function?

$$f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Evaluating this limit was a nuisance, even for fairly simple functions f. Fortunately, this definition rarely needs to be used any more, because the definition was USED to develop tools that allow us to thereafter BYPASS the definition; tools such as the simple power rule, chain rule, product rule, and quotient rule. Similarly, the Fundamental Theorem of Integral Calculus gives a convenient tool for computing the definite integral of f, whenever we can get our hands on an antiderivative of f. Here's a precise statement of the fundamental theorem:

Fundamental Theorem of Integral Calculus

Let f be continuous on $[a,b]$. If F is *any* antiderivative for f on $[a,b]$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

★★ *antiderivative of f on $[a,b]$*

For F to be an antiderivative of f on $[a,b]$, not only must $F'(x) = f(x)$ for all $x \in (a,b)$, but F must also be continuous on the closed interval $[a,b]$. In particular, F must 'behave properly' at the endpoints a and b.

some notation used in connection with the definite integral; $F(x)\big|_a^b$

Let F be an antiderivative of f. The following notation is used in connection with evaluating the definite integral:

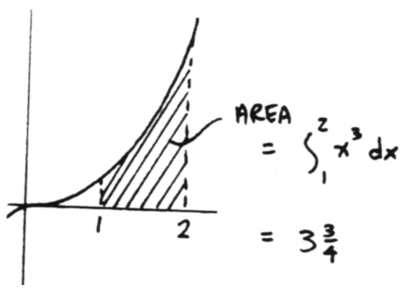
$$\begin{aligned}\int_a^b f(x)\,dx &= F(x)\big|_a^b \\ &= F(b) - F(a)\end{aligned}$$

That is, the notation $F(x)\big|_a^b$ is used to represent the operation of evaluating the antiderivative F at b, evaluating it at a, and then subtracting these two numbers, as illustrated in the next example.

EXAMPLE

Problem: Compute the definite integral:

$$\int_1^2 x^3\,dx$$



Solution: Find an antiderivative of x^3, and use the Fundamental Theorem. The simplest antiderivative of $f(x) = x^3$ is $F(x) = \frac{x^4}{4}$. Using this antiderivative to evaluate the definite integral yields:

$$\begin{aligned}\int_1^2 x^3\,dx &= \frac{x^4}{4}\Big|_1^2 \\ &= \frac{(2)^4}{\mathbf{4}} - \frac{(1)^4}{\mathbf{4}} \\ &= 4 - \frac{1}{4} = 3\frac{3}{4}\end{aligned}$$

Since x^3 is positive on $[1,2]$, the number $3\frac{3}{4}$ gives the area under the graph of f on $[1,2]$.

factor constants out first

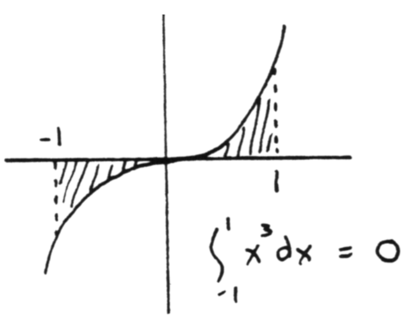
Note that the constant $\frac{1}{4}$ appears in **both terms** (shown bold above) in the evaluation process. It is usually easiest to factor this constant out first, and more simply write:

$$\begin{aligned}\int_1^2 x^3\,dx = \frac{x^4}{4}\Big|_1^2 &= \frac{1}{4}(2^4 - 1^4) \\ &= \frac{1}{4}(16-1) = \frac{15}{4} = 3\frac{3}{4}\end{aligned}$$

EXAMPLE

Problem: Compute the definite integral:

$$\int_{-1}^1 x^3\,dx$$



Solution: Find an antiderivative of x^3, and use the Fundamental Theorem. Observe that, this time, x^3 is NOT positive over the entire interval of integration. Applying the fundamental theorem:

$$\begin{aligned}\int_{-1}^1 x^3\,dx = \frac{x^4}{4}\Big|_{-1}^1 &= \frac{1}{4}(1^4 - (-1)^4) \\ &= \frac{1}{4}(1-1) = \frac{1}{4}(0) = 0\end{aligned}$$

Momentarily, it will be made clear why the answer is 0. ♣ Any speculation?

EXERCISE 2

Evaluate the following definite integrals. Be sure to write complete mathematical sentences. When possible, interpret your answer in terms of area.

♣ 1. $\int_0^1 x^5\,dx$

♣ 2. $\int_0^4 e^x\,dx$

♣ 3. $\int_{-4}^0 e^x\,dx$

♣ 4. $\int_1^2 \frac{1}{x}\,dx$

♣ 5. $\int_{1/2}^2 \frac{1}{x}\,dx$

dummy variable of integration

Since the definite integral is a *number*, the variable of integration is irrelevant. That is, once $\int_a^b f(x)\,dx$ is evaluated, the letter 'x' is gone. Any letter may be used; for example, one can write

$$\int_a^b f(x)\,dx \quad\text{or}\quad \int_a^b f(t)\,dt \quad\text{or}\quad \int_a^b f(\omega)\,d\omega\ ;$$

they are all equal. Just be sure to carry this same letter through your computations; for example:

$$\int_1^2 t^3\,dt = \frac{t^4}{4}\Big|_1^2 = \frac{1}{4}(2^4 - 1^4) = 3\frac{3}{4}$$

The letter used in $\int_a^b f(x)\,dx$ is called the *dummy variable of integration.*

EXERCISE 3

♣ Suppose you KNOW $\int_a^b f(x)\,dx$. Then, do you know $\int_a^b f(t)\,dt$? How about $\int_a^b f(s)\,ds$? How about $\int_c^d f(x)\,dx$? How about $\int_a^b g(x)\,dx$?

To begin to better understand the definite integral, some properties that it satisfies are stated next.

Properties of the Definite Integral

Suppose that f is continuous on $[a,b]$.

linearity

For all constants k :

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

That is, constants can be pulled out of the definite integral.

Also:

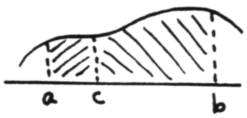
$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

That is, the integral of a sum is the sum of the integrals.

Together, these two properties are referred to as the *linearity of the (definite) integral.*

Properties of the Definite Integral

additivity



There are also properties that relate to the limits of integration. Again suppose that f is continuous on $[a, b]$.

For any $c \in (a, b)$:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

That is, to integrate from a to b, one can choose any number c between a and b, and integrate instead in two pieces: from a to c, and then from c to b.

This property is referred to as the *additivity of the integral*.

Finally, for all real numbers a and b:

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

The last property says that if you integrate 'backwards', you must introduce a minus sign.

If F is an antiderivative of f, then the fundamental theorem can be used to find the definite integral, and we see that:

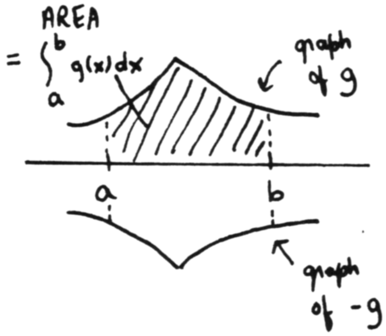
$$\begin{aligned}\int_b^a f(x)\,dx &= F(a) - F(b)\\ &= -(F(b) - F(a))\\ &= -\int_a^b f(x)dx\end{aligned}$$

This explains the last property above.

the definite integral treats area under the x-axis as negative

Let's investigate the property:

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



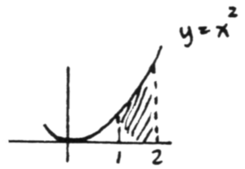
This property shows that if a *negative* function is integrated, then a *negative* number will be obtained. The *magnitude* of this negative number corresponds to the magnitude of the area beneath the x-axis.

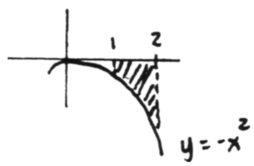
To see this, suppose that g is positive on $[a, b]$. Then, $-g$ is negative on $[a, b]$, and:

$$\int_a^b (-g(x))\,dx = -\int_a^b g(x)\,dx$$

The graphs of g and $-g$ are symmetric about the x-axis; $\int_a^b g(x)\,dx$ gives the area under the graph of g. Thus, *the definite integral treats area under the x-axis as negative.*

EXAMPLE





$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2 = \frac{1}{3}(2^3 - 1^3) = \frac{7}{3}$$

and

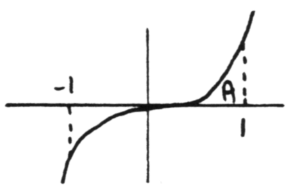
$$\int_1^2 -x^2\,dx = -\int_1^2 x^2\,dx = -\frac{7}{3}$$

Here, since the function $-x^2$ is negative over the entire interval $[1,2]$, the result $-\frac{7}{3}$ is interpreted as follows:

- The magnitude of the result, $|-\frac{7}{3}| = \frac{7}{3}$, indicates that there is $\frac{7}{3}$ units of area trapped between the graph of $-x^2$ and the x-axis.
- The fact that the answer $-\frac{7}{3}$ is *negative* indicates that this area lies *beneath* the x-axis.

Caution!!
$\int_a^b f(x)\,dx = 0$
does **not** *imply that*
$f(x) = 0$

If one integrates over an interval $[a,b]$ on which the area trapped between the graph of f and lying *above* the x-axis is the same as that area *below* the x-axis, then the additivity of the integral shows that the definite integral will have value 0.

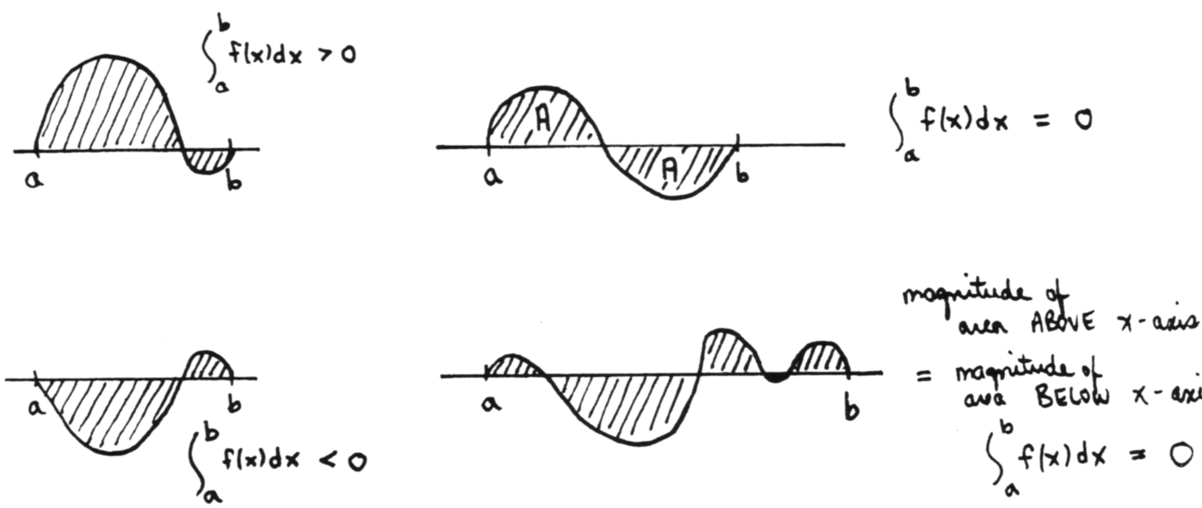


For example, it was seen earlier that $\int_{-1}^1 x^3\,dx = 0$. This is because, by additivity:

$$\begin{aligned}\int_{-1}^1 x^3\,dx &= \int_{-1}^0 x^3\,dx + \int_0^1 x^3\,dx \\ &= (-A) + (A) = 0\ ,\end{aligned}$$

where A represents the magnitude of the area trapped between the graph of x^3 and the x-axis on $[0,1]$.

So, *just because* $\displaystyle\int_a^b f(x)\,dx = 0$ *does not necessarily mean that* $f(x) = 0$ *on* $[a,b]$. Instead, it means that the area trapped between the graph of f and lying *above* the x-axis, is the same as the area trapped between the graph of f and lying *below* the x-axis, on the interval $[a,b]$.



EXERCISE 4

The graph of a function f is shown below, and certain areas are labeled. Based on this information, evaluate the following integrals, if possible. If this is not possible based on the given information, so state.

♣ 1. $\int_{-3}^{-2} f(x)\,dx$

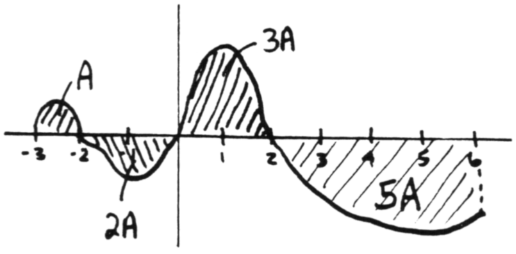
♣ 2. $\int_{-3}^{0} f(t)\,dt$

♣ 3. $\int_{-3}^{2} f(s)\,ds$

♣ 4. $\int_{0}^{5} f(x)\,dx$

♣ 5. $\int_{-2}^{2} f(t)\,dt$

♣ 6. $\int_{-3}^{-1} f(y)\,dy$



This section is concluded with some examples that illustrate how the properties of the definite integral can be used to help in its evaluation.

EXAMPLE

Problem: Evaluate $\int_0^1 (x^2 - 2x + 3)\,dx$.

Solution: By linearity:

$$\begin{aligned}
\int_0^1 (x^2 - 2x + 3)\,dx &= \int_0^1 x^2\,dx - 2\int_0^1 x\,dx + 3\int_0^1 (1)\,dx \\
&= \frac{x^3}{3}\Big|_0^1 - 2\cdot\frac{x^2}{2}\Big|_0^1 + 3\cdot x\Big|_0^1 \\
&= \frac{1}{3}(1-0) - \frac{2}{2}(1-0) + 3(1-0) \\
&= \frac{1}{3} - 1 + 3 = 2\frac{1}{3}
\end{aligned}$$

The solution is usually written down in a much more abbreviated form:

$$\begin{aligned}
\int_0^1 (x^2 - 2x + 3)\,dx &= \left(\frac{x^3}{3} - 2\cdot\frac{x^2}{2} + 3x\right)\Big|_0^1 \\
&= \left(\frac{1}{3} - 1 + 3\right) - (0) = 2\frac{1}{3}
\end{aligned}$$

EXAMPLE

find the indefinite integral first; then use any antiderivative to find the definite integral

When the integrand in a definite integral problem is complicated, some people prefer to *first* solve the companion indefinite integral, and then *use* any antiderivative to find the definite integral. This prevents having to 'carry around' the limits of integration.

Problem: Find the area under the graph of $\frac{1}{3x+1}$ on $[0, 2]$.

Solution: It is not necessary to graph $\frac{1}{3x+1}$; it is only necessary to recognize that whenever $x \in [0, 2]$, $\frac{1}{3x+1} > 0$. Thus, the graph lies entirely above the x-axis on this interval, and the desired area is given by the definite integral:

$$\int_0^2 \frac{1}{3x+1}\,dx$$

In a future section, we will discuss how to use the technique of substitution *directly* with definite integrals. For now, find an antiderivative by first solving the companion indefinite integral problem:

$u = 3x + 1$
$du = 3dx$

$$\begin{aligned}\int \frac{1}{3x+1}\,dx &= \frac{1}{3}\int \frac{1}{3x+1}3\,dx = \frac{1}{3}\int \frac{1}{u}\,du \\ &= \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|3x+1| + C\end{aligned}$$

Use the simplest antiderivative to evaluate the desired definite integral:

$$\int_0^2 \frac{1}{3x+1}\,dx = \frac{1}{3}\ln|3x+1|\,\Big|_0^2 = \frac{1}{3}(\ln 7 - \ln 1) = \frac{1}{3}\ln 7 \approx 0.65$$

One more time!

Since variety is the spice of life, the previous problem is solved in a different way:

$$\begin{aligned}\int_0^2 \frac{1}{3x+1}\,dx &= \int_0^2 \frac{1}{3(x+\frac{1}{3})}\,dx = \frac{1}{3}\int_0^2 \frac{1}{x+\frac{1}{3}}\,dx \\ &= \frac{1}{3}\ln|x+\frac{1}{3}|\,\Big|_0^2 = \frac{1}{3}(\ln\frac{7}{3} - \ln\frac{1}{3}) \\ &= \frac{1}{3}(\ln\frac{7/3}{1/3}) = \frac{1}{3}(\ln 7) \approx 0.65\end{aligned}$$

EXERCISE 5 ♣ Evaluate $\int_0^1 \frac{1}{5x+1}\,dx$ in two ways.

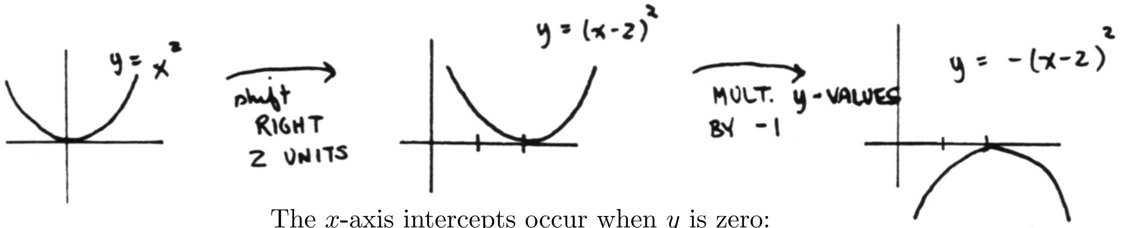
EXAMPLE

Problem: Determine the area of the region bounded by the graph of

$$y = -(x-2)^2 + 1$$

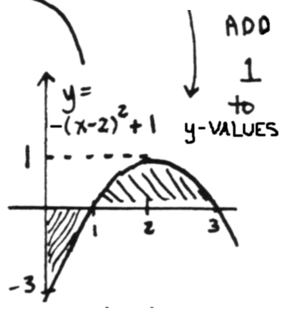
and the x-axis on the interval $[0, 3]$.

Solution: A quick sketch is easy to get and helpful. View y as being 'built up' as follows:



The x-axis intercepts occur when y is zero:

$$\begin{aligned} y = 0 \quad &\iff \quad -(x-2)^2 + 1 = 0 \\ &\iff \quad (x-2)^2 = 1 \\ &\iff \quad |x-2| = 1 \\ &\iff \quad x = 3 \quad \text{or} \quad x = 1 \end{aligned}$$



Here we used the facts that:

- For all real numbers x, $\sqrt{x^2} = |x|$; and
- $|x-2|$ tells us how far x is from 2. Thus, $|x-2| = 1$ is true exactly when x is a number whose distance from 2 equals 1.

Now, to find the desired area, we MUST integrate in two pieces:

$$\begin{aligned} \int_0^1 \big(-(x-2)^2 + 1\big)\,dx &= -\frac{(x-2)^3}{3} + x \,\Big|_0^1 \\ &= \big(-\frac{(1-2)^3}{3} + 1\big) - \big(-\frac{(0-2)^3}{3} + 0\big) \\ &= \frac{4}{3} - \frac{8}{3} = -\frac{4}{3} \end{aligned}$$

The answer is negative, because the area is beneath the x-axis.

Also:

$$\int_1^3 \big(-(x-2)^2 + 1\big)\,dx = \ldots = \frac{4}{3}$$

The desired area is therefore:

$$\frac{4}{3} + \frac{4}{3} = \frac{8}{3}$$

EXERCISE 6

♣ What would have happened if, in the previous problem, you had tried to compute the desired area by finding $\int_0^3 \big(-(x-2)^2 + 1\big)\,dx$? Evaluate this integral to confirm your answer.

EXERCISE 7

♣ Determine the area of the region bounded by the graph of $y = (x+2)^2 - 1$ on the interval $[-3, 0]$. Make a sketch showing the area that you are finding.

QUICK QUIZ
sample questions

1. In a few words, explain why there is such a similar appearance between the indefinite integral $\int f(x)\,dx$ and the definite integral $\int_a^b f(x)\,dx$.
2. Give a precise statement of the Fundamental Theorem of Integral Calculus.
3. What does the notation $F(x)\big|_a^b$ mean, when used in the context of evaluating definite integrals?
4. Compute: $\int_{-1}^{2} x^2\,dx$
5. Show that: $\int_{-1}^{1} x^3\,dx = 0$
   Interpret your answer in terms of area.

KEYWORDS
for this section

Notation for the definite integral, upper and lower limits of integration, comparing the definite and indefinite integrals, the Fundamental Theorem of Integral Calculus, the notation $F(x)\big|_a^b$, dummy variable, properties of the definite integral, linearity, additivity, integrating backwards introduces a minus sign, the definite integral treats area under the x-axis as negative.

END-OF-SECTION EXERCISES

Evaluate the following integrals. Use any appropriate methods. Be sure to write complete mathematical sentences.

1. $\displaystyle\int_0^2 \frac{3}{2}x^4\,dx$
2. $\displaystyle\int_1^8 t^{1/3}\,dt$
3. $\displaystyle\int_{-1}^1 (2x-3)\,dx$
4. $\displaystyle\int_0^1 (ax+b)\,dx$
5. $\displaystyle\int_0^1 \frac{x^2}{1+x^3}\,dx$
6. $\displaystyle\int_{\ln 2}^{\ln 3} e^{2t}\,dt$

Find the area bounded by the graph of the given function and the x-axis on the stated interval. Make a sketch showing the area that you are finding. You may have to evaluate more than one integral to obtain your final answer.

7. $f(x) = 1 + e^x;\ [0,2]$
8. $f(x) = (x-1)(x+3);\ [-2,2]$
9. $f(x) = 2x^2 + 5x - 3;\ [0,2]$