7.6 Finding the Volume of a Solid of Revolution—Disks

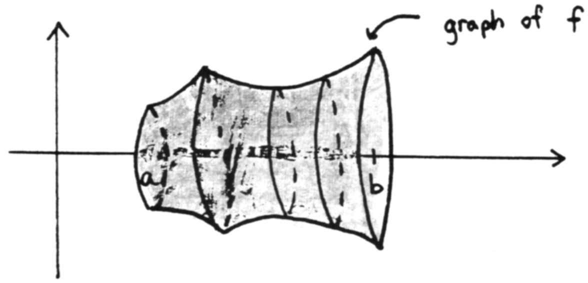
Introduction

Again, in this section, intuition gained from the definition of the definite integral helps to motivate some useful formulas for finding the volume of solids of revolution. Keep in mind that strict derivations of these formulas would require partitioning and investigating Riemann sums.

generating a solid of revolution;

revolving about the x-axis

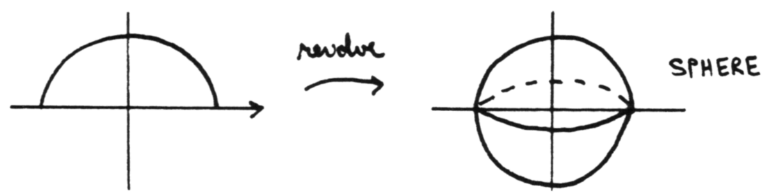
Let f be continuous on an interval $[a, b]$. If the area between the graph of f and the x-axis on $[a, b]$ is rotated about the x-axis, then a *solid of revolution* is generated. Our goal is to use calculus to find the volume of this solid of revolution.



EXAMPLE

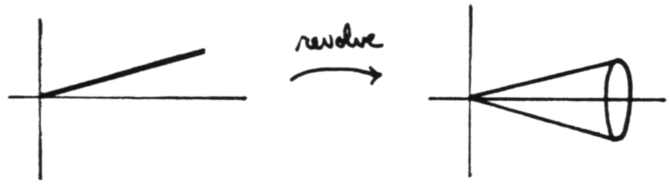
For example, consider the upper-half-circle shown below. When this graph is rotated about the x-axis, a sphere results.

The phrase 'this graph is rotated about the x-axis' is shorthand for the more correct phrase, 'the area between the graph and the x-axis is rotated about the x-axis'.



EXAMPLE

If the line shown below is revolved about the x-axis, a right circular cone is obtained.



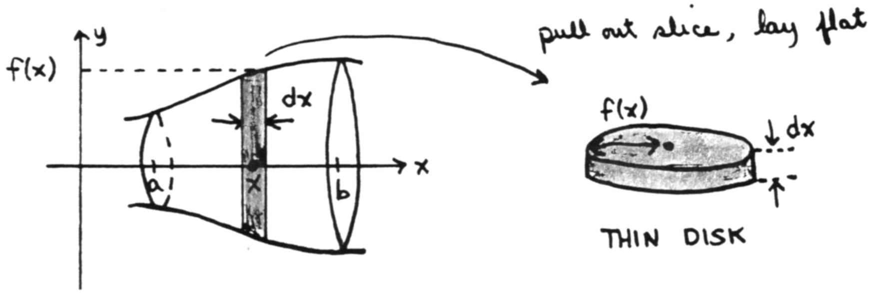
These two examples will be used to 'test' our formula, after its derivation.

motivational derivation of the formula;

Let f be continuous on $[a,b]$. For the moment, suppose that f is nonnegative, so that its graph lies above the x-axis. (This restriction will be removed momentarily.)

a typical slice of the solid

Revolve the graph about the x-axis. Let's investigate a *typical infinitesimal slice* of the resulting solid of revolution.



the slice is a disk with volume $\pi(f(x))^2\,dx$

Choose a value x between a and b. Imagine holding a saw, perpendicular to the xy-plane, and cutting a thin slice from the desired solid at this value x. Call the thickness of this slice dx, and think of dx as representing an *infinitesimally small* piece of the x-axis. Pull this slice out and lay it down. It looks like a disk! (Consequently, this technique is often referred to as *the disk method.*) The radius of the disk is $f(x)$, and hence its volume is:

$$(\text{area of circle})(\text{thickness}) = \pi(f(x))^2 \cdot dx$$

Now, use integration to 'sum' these slices, as x travels from a to b:

$$\text{desired volume} = \int_a^b \pi\big(f(x)\big)^2\,dx$$

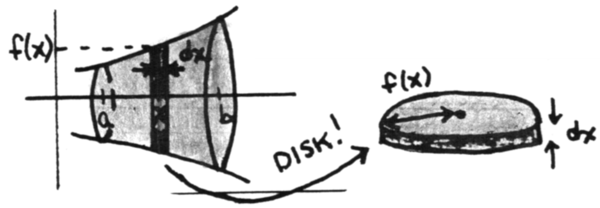
f can be negative

Observe that if f is negative, the radius of the resulting slice is $|f(x)|$, but, (since this radius is squared in finding the area of the circle), the volume of the typical slice is still $\pi(f(x))^2\,dx$. Thus, the formula holds for all continuous functions f.

The result is summarized below:

DISK METHOD

Let f be continuous on $[a,b]$. If the area between the graph of f and the x-axis on the interval $[a,b]$ is revolved about the x-axis, then the volume of the resulting solid of revolution is:



$$\int_a^b \pi(f(x))^2\,dx$$

EXAMPLE

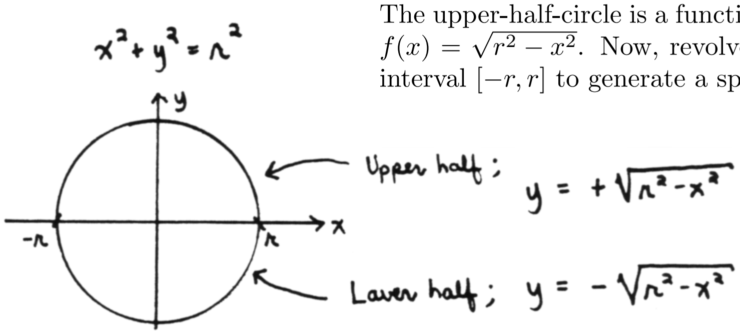
testing the formula; finding the volume of a sphere

Problem: Recall that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. Derive this formula by investigating an appropriate integral.

Solution: The circle of radius r with center at the origin is the set of all points (x, y) satisfying the equation $x^2 + y^2 = r^2$; solving for y yields:

$$y^2 = r^2 - x^2 \quad \iff \quad |y| = \sqrt{r^2 - x^2} \quad \iff \quad y = \pm\sqrt{r^2 - x^2}$$

The upper-half-circle is a function; its equation is obtained by using the + sign: $f(x) = \sqrt{r^2 - x^2}$. Now, revolve this upper-half-circle about the x-axis on the interval $[-r, r]$ to generate a sphere of radius r.



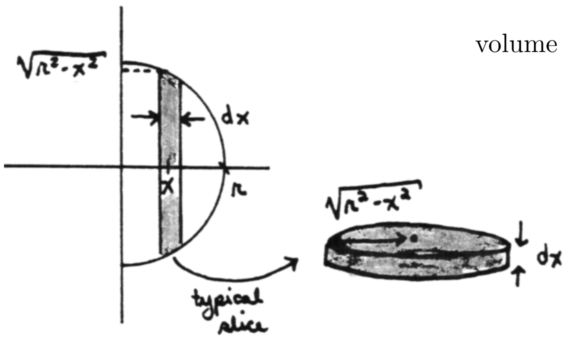
take advantage of symmetry

To cut down on the algebra, we can take advantage of symmetry and find the volume of the half sphere over the interval $[0, r]$; doubling this yields the desired result.

A typical infinitesimal slice of the desired solid at $x \in [0, r]$ has volume:

$$\pi(\sqrt{r^2 - x^2})^2\,dx$$

Then, using calculus to 'sum' these slices yields:



$$\begin{aligned}
\text{volume of sphere} &= 2 \cdot \int_0^r \pi\big(\sqrt{r^2 - x^2}\big)^2\,dx \\
&= 2\pi \int_0^r (r^2 - x^2)\,dx \\
&= 2\pi(r^2 x - \frac{x^3}{3})\Big|_0^r \\
&= 2\pi\big[(r^3 - \frac{r^3}{3}) - 0\big] = 2\pi\frac{2r^3}{3} \\
&= \frac{4}{3}\pi r^3
\end{aligned}$$

The expected result is indeed obtained.

EXERCISE 1	♣ Repeat the previous exercise, without looking at the text.

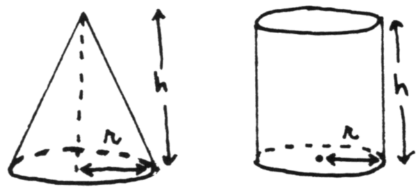
EXAMPLE

testing the formula; finding the volume of a right circular cone

Problem: Recall that the volume of a right circular cone of height h and base radius r is:

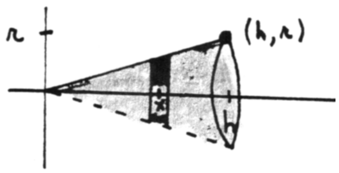
$$\frac{1}{3}\pi r^2 h$$

Thus, three such cones would completely fill the cylinder of height h and base radius r. Derive the formula for the volume of a right circular cone by investigating an appropriate integral.



Solution #1

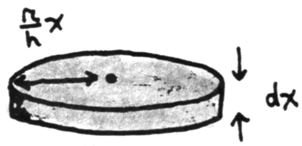
Solution #1: First, find the equation of the line passing through the point $(0,0)$ and (h,r); it has slope $\frac{r}{h}$ and passes through the origin, so has equation $y = \frac{r}{h}x$.



Revolving this graph about the x-axis on $[0,h]$ yields the desired solid.

An infinitesimal slice at $x \in [0,h]$ has volume

$$\pi(\frac{r}{h}x)^2\,dx$$



and integration over $[0,h]$ yields

$$\begin{aligned}
\text{desired volume} &= \int_0^h \pi(\frac{r}{h}x)^2\,dx \\
&= \pi\frac{r^2}{h^2}\int_0^h x^2\,dx \\
&= \pi\frac{r^2}{h^2}\cdot\frac{x^3}{3}\Big|_0^h \\
&= \pi\frac{r^2}{h^2}\big[(\frac{h^3}{3})-0\big] \\
&= \frac{1}{3}\pi r^2 h\ ,
\end{aligned}$$

which is of course the anticipated result.

EXERCISE 2

♣ 1. Repeat the previous example, without looking at the text.

♣ 2. Use calculus to find the volume of a cylinder of height h and radius r, by investigating an appropriate integral.

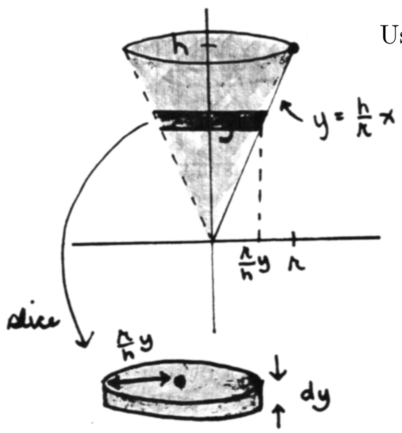
Solution #2

Solution #2. The 'disk formula' can also be applied to find the volume of a solid that results from revolution about the y-axis. It is only necessary that a typical 'slice' be a disk. This is illustrated in the next, alternate, derivation of the volume of a right circular cone.

This time, generate the cone by revolving the line $y = \frac{h}{r}x$ about the y-axis.

Observe that $y = \frac{h}{r}x \iff x = \frac{r}{h}y$. Make a thin (thickness dy) horizontal slice at a typical distance y, where $y \in [0, h]$. The volume of this slice is:

$$\pi(\frac{r}{h}y)^2 \cdot dy$$



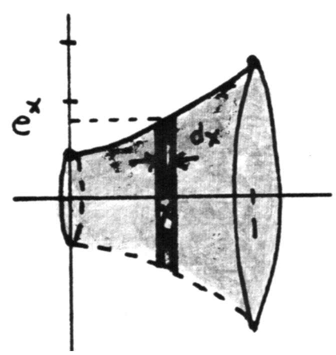
Using calculus to sum these slices as y varies from 0 to h yields:

$$\begin{aligned}\text{desired volume} &= \int_0^h \pi(\frac{r}{h}y)^2\,dy \\ &= \cdots = \frac{\pi r^2 h}{3}\end{aligned}$$

EXAMPLE

Problem: Revolve the graph of e^x about the x-axis on $[0, 1]$. Find the volume of the resulting solid of revolution.

Solution:



$$\begin{aligned}\text{desired volume} &= \int_0^1 \pi(e^x)^2\,dx \\ &= \pi\int_0^1 e^{2x}\,dx \\ &= \pi \cdot \frac{1}{2}e^{2x}\big|_0^1 \\ &= \frac{\pi}{2}(e^2 - 1) \approx 10.036\end{aligned}$$

EXAMPLE

Problem: Take the graph of x^2 on $[0, 2]$ and revolve it about the y-axis. Find the volume of the resulting solid of revolution.

Solution: Observe that the top of the desired solid is at $y = 4$, and the bottom is at $y = 0$.

Make a thin (thickness dy) horizontal slice through the solid at distance $y \in [0, 4]$. For $y \geq 0$,

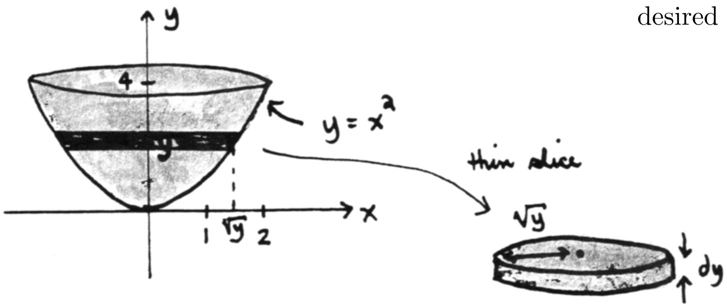
$$y = x^2 \quad \iff \quad x = \pm\sqrt{y}\ ,$$

so the radius of the thin slice is $\sqrt{y}$, and has volume:

$$\pi(\sqrt{y})^2\, dy$$

Using calculus to 'sum' the disks as y goes from 0 to 4 yields:

$$\begin{aligned} \text{desired volume} &= \int_0^4 \pi(\sqrt{y})^2\, dy \\ &= \pi \int_0^4 y\, dy \\ &= \pi \cdot \frac{y^2}{2}\Big|_0^4 \\ &= \frac{\pi}{2}(16 - 0) = 8\pi \end{aligned}$$

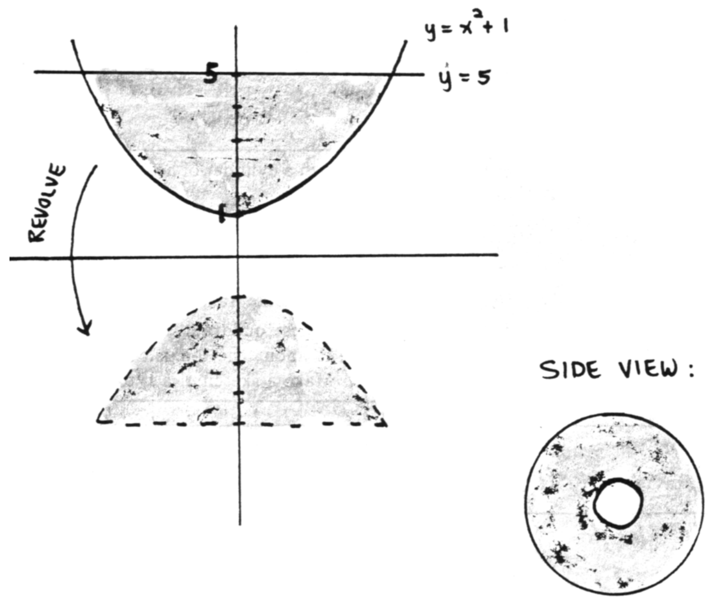


EXAMPLE

a solid with a hole

Problem: Find the volume of the solid generated by taking the region bounded by $y = 5$ and $y = x^2 + 1$, and revolving it about the x-axis.

Solution: The resulting solid of revolution has a hole in it. Note that the graphs $y = 5$ and $y = x^2 + 1$ intersect at values of x for which $5 = x^2 + 1$; solving this equation gives $x = \pm 2$. This problem will be solved in two different ways.

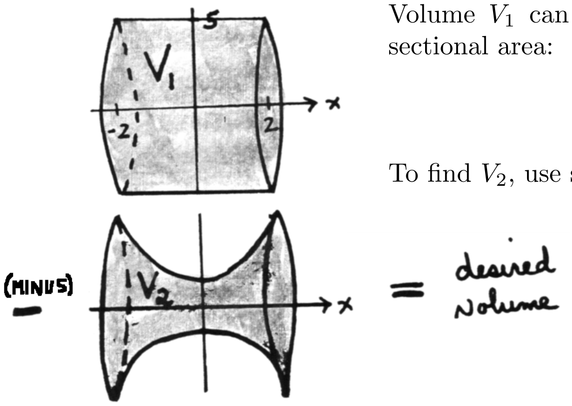


Approach #1;
view the desired volume as a difference of volumes

Approach #1. The desired volume can be viewed as a difference of volumes:
Revolve $y = 5$ about the x-axis on $[-2, 2]$; call this volume V_1 .
Revolve $y = x^2 + 1$ about the x-axis on $[-2, 2]$; call this volume V_2 .
The desired volume is $V_1 - V_2$.

Volume V_1 can be found *without* calculus, since it is has a constant cross-sectional area:

$$V_1 = (\text{area of circle})(\text{height}) = \pi(5)^2 \cdot 4 = 100\pi$$



To find V_2, use symmetry and calculus:

$$\begin{aligned} V_2 &= 2\int_0^2 \pi(x^2+1)^2\,dx \\ &= 2\pi\int_0^2 (x^4+2x^2+1)\,dx \\ &= \cdots = \frac{412\pi}{15} \end{aligned}$$

Thus, the desired volume is $V_1 - V_2 = 100\pi - \frac{412\pi}{15} = 72.5\bar{3}\pi$.

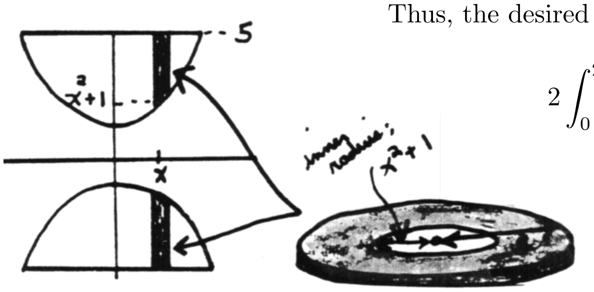
Approach #2;
look at an infinitesimal slice

Approach #2. This time, let's investigate a typical thin slice of the desired solid, at a distance $x \in [0, 2]$. It is shaped like a donut, and has volume:

$$\pi(5)^2\,dx - \pi(x^2+1)^2\,dx = \pi[5^2 - (x^2+1)^2]\,dx$$

Thus, the desired volume is (again using symmetry):

$$\begin{aligned} 2\int_0^2 \pi[5^2 - (x^2+1)^2]\,dx &= 2\pi\int_0^2 24 - x^4 - 2x^2\,dx \\ &= \cdots = 72.5\bar{3}\pi \end{aligned}$$



QUICK QUIZ *sample questions*	1. Show two ways in which a cylinder of height h and radius r can be generated as a solid of revolution. 2. Show two ways in which a right circular cone of height h and base radius r can be generated as a solid of revolution. 3. Revolve the graph of x^2 about the x-axis on $[0, 1]$. Find the volume of the resulting solid of revolution. Make a sketch of a typical 'slice'. 4. Take the area in the first quadrant bounded by $y = x^2$, the y-axis, and $y = 1$, and revolve it about the y-axis. Find the volume of the resulting solid of revolution. Make a sketch of a typical 'slice'.

KEYWORDS *for this section*	*Generating a solid of revolution by revolving about the x-axis; what is the volume of a typical thin slice? The disk method, revolving about the y-axis, a solid with a hole.*

END-OF-SECTION EXERCISES

♣ Revolve each region described below about the x-axis. Find the volume of the resulting solid of revolution. Be sure to write complete mathematical sentences. Make a rough sketch of the solid under investigation.

1. Bounded by: $y = 2x$, $x = 0$, $x = 1$, and the x-axis
2. Bounded by: $y = x^3$, $x = 1$, $x = 2$, and the x-axis
3. Bounded by: $y = \frac{1}{x}$, $x = 1$, $x = 2$, and the x-axis
4. Bounded by: $y = |x|$, $x = -1$, $x = 1$, and the x-axis
5. Bounded by: $y = \sqrt{x}$, $x = 0$, $x = 4$, and the x-axis
6. Bounded by: $y = e^x + 1$, $x = 0$, $x = 1$, and the x-axis
7. In the first quadrant, bounded by: $y = x^2$, $y = 0$, $y = 4$, and the y-axis
   (A typical slice will have a hole—be careful.)
8. Bounded by: $y = x^3$, $y = 0$, $y = 8$, and the y-axis
   (A typical slice will have a hole—be careful.)

Revolve each region described below about the y-axis. Find the volume of the resulting solid of revolution. Be sure to write complete mathematical sentences. Make a rough sketch of the solid under investigation.

9. Bounded by: $y = x$, $y = 0$, $y = 2$, and the y-axis
10. Bounded by: $y = 2x$, $y = 1$, $y = 3$, and the y-axis
11. Bounded by: $y = \frac{1}{x}$, $y = 1$, $y = 2$, and $x = \frac{1}{2}$
    (The resulting solid will have a hole—be careful.)