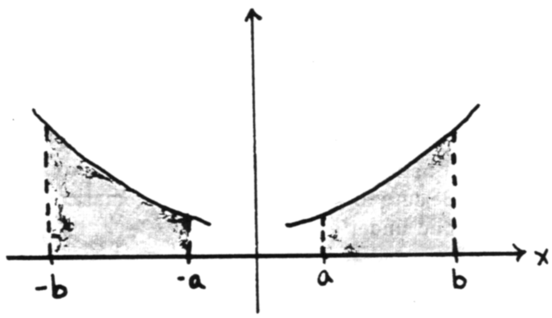
7.7 Finding the Volume of a Solid Of Revolution—Shells

generating a volume of revolution;

revolving about the y-axis

Let f be continuous and nonnegative on $[a,b]$. Take the area bounded by the graph of f and the x-axis on the interval $[a,b]$, and revolve it about the y-axis.

In some instances, the volume of the resulting solid of revolution can be found by looking at disks (or disks with holes) that are sliced *horizontally*, that is, perpendicular to the y-axis. However, it is shown in this section that there is a more natural way to view the resulting solid in this case; as being built up from *thin shells.*



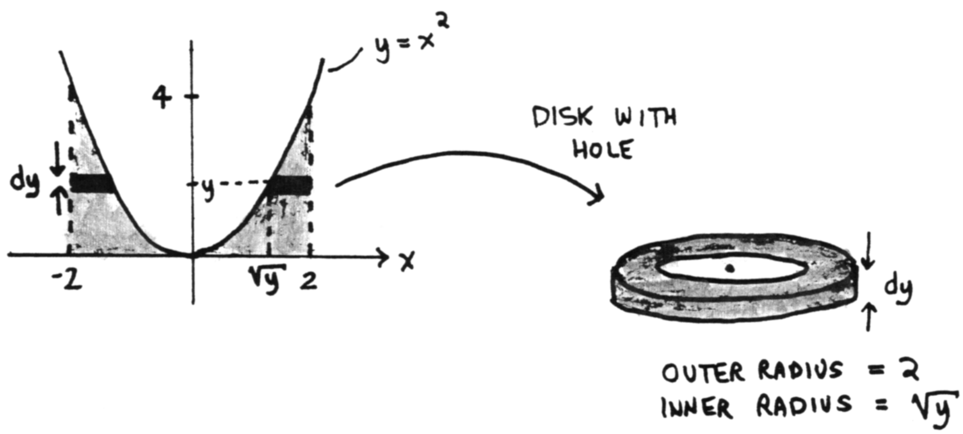
EXAMPLE

horizontal disks with holes

Problem: Revolve the area bounded by $f(x) = x^2$ and the x-axis on $[0,2]$ about the y-axis. Find the volume of the resulting solid of revolution, using horizontal disks.

Solution: The method discussed here was introduced in the previous section. This is a review problem.

When $x = 2$, $y = 2^2 = 4$. Let y denote a typical value in $[0,4]$, and cut a thin horizontal slice (thickness dy) from the desired volume at this value of y. As usual, view dy as an *infinitesimally small* piece of the y-axis. The slice is a disk with a hole (a donut); what is its volume?



get an expression for x in terms of y

Given y, it is necessary to know the corresponding value of x (since the x-value of the point determines the inner radius of the donut). That is, a formula for x in terms of y is needed. Solving $y = x^2$ for x yields:

$$y = x^2 \quad \iff \quad |x| = \sqrt{y} \quad \iff \quad x = \pm\sqrt{y}$$

Two answers are obtained, since, viewed from the y-axis, the curve is *not* a function of y. The positive number $+\sqrt{y}$ is chosen to give the inner radius of the donut.

The volume of this slice is found by first getting the volume of the slice when it doesn't have a hole, and then subtracting off the volume of the hole:

$$\pi(2)^2\,dy - \pi(\sqrt{y})^2\,dy \;=\; \pi(4-y)\,dy$$

Then, 'sum' these slices, as y travels from 0 to 4:

$$\begin{aligned}\text{desired volume} &= \int_0^4 \pi(4-y)\,dy \\ &= \pi(4y - \frac{y^2}{2})\Big|_0^4 \\ &= \pi(16 - \frac{16}{2}) = 8\pi\end{aligned}$$

EXERCISE 1	♣ Problem: Revolve the area bounded by $f(x) = x^3$ and the x-axis on $[0,2]$ about the y-axis. Find the volume of the resulting solid of revolution, by using horizontal disks. Make a sketch of the volume that you are finding. Also make a sketch of a typical 'slice'.

disadvantages of the disk approach in this setting

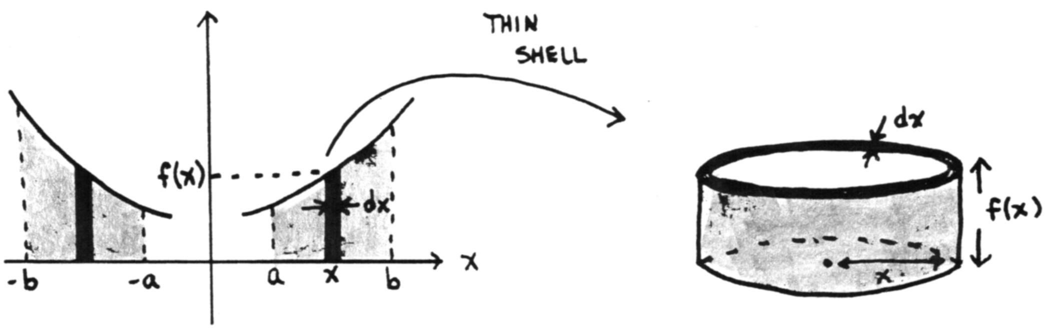
The previous approach was 'hard' in two ways:

- It was necessary to solve for x in terms of y. This is unnatural, since although y *is* a function of x, x may *not* be a function of y.
- The typical slice was not a simple disk, but a disk with a hole, which is more difficult to work with.

These disadvantages are overcome by viewing the volume in a different way, as discussed below.

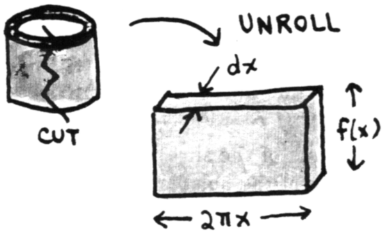
the shell method

Let f be continuous and nonnegative on $[a,b]$. Take the area bounded by the graph of f and the x-axis on $[a,b]$, and revolve it about the y-axis. Take a 'donut cutter' of radius x (where x is a number between a and b), and, coming down from the top, punch a thin shell (thickness dx) from the solid of revolution.



the shell has volume $2\pi x f(x)\,dx$

To calculate the volume of this thin shell, observe first that its circumference is $2\pi(\text{radius}) = 2\pi x$, and its height is $f(x)$. Cut the shell and unroll it. The volume is now easy to calculate:



$$(\text{width})(\text{height})(\text{thickness}) = (2\pi x)f(x)(dx)$$

Summing the volumes of these shells as x travels from a to b yields the desired volume of revolution:

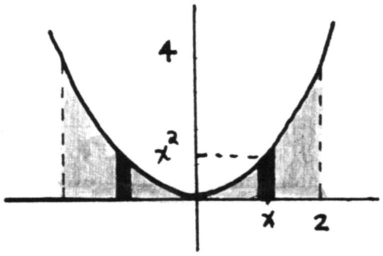
$$(\text{desired volume}) = \int_a^b 2\pi x\, f(x)\, dx$$

Remember that a rigorous derivation of this formula would require partitioning, and looking at Riemann sums.

EXAMPLE

Problem: Revolve the area bounded by $f(x) = x^2$ and the x-axis on $[0, 2]$ about the y-axis. Using shells, find the volume of the resulting solid of revolution.

Solution: The solution is now much easier than when the volume was viewed as being 'built up' from horizontal disks:



$$\begin{aligned}(\text{desired volume}) &= \int_0^2 2\pi x\,(x^2)\,dx \\ &= 2\pi \int_0^2 x^3\,dx \\ &= 2\pi\ \frac{x^4}{4}\Big|_0^2 \\ &= \frac{\pi}{2}(16-0) \\ &= 8\pi\end{aligned}$$

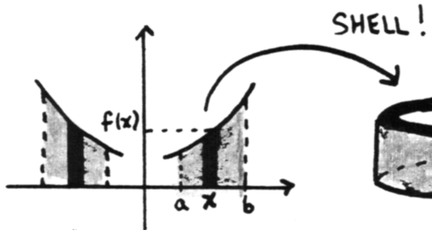
Note that you only integrate from 0 to 2; yet the volume being found extends from -2 to 2. (♣ Why is this?)

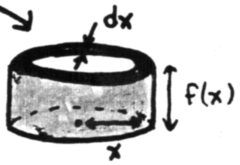
The result is summarized below.

SHELL METHOD

Let f be continuous and nonnegative on $[a, b]$. If the area between the graph of f and the x-axis on $[a, b]$ is revolved about the y-axis, then the volume of the resulting solid of revolution is:

$$\int_a^b 2\pi x\, f(x)\, dx$$



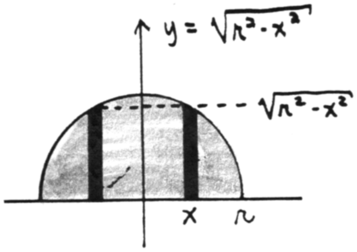


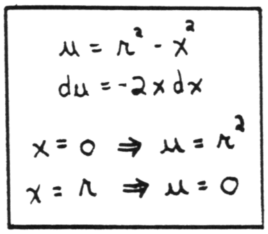
EXAMPLE

finding the volume of a sphere, using shells

Problem: Derive the formula $V = \frac{4}{3}\pi r^3$ for the volume of a sphere of radius r, using shells.

Solution: As shown in the previous section, the upper-half circle of radius r has equation $y = \sqrt{r^2 - x^2}$. Take the area bounded by this curve and the x-axis on $[0, r]$ and revolve it about the y-axis. Double this volume to obtain the desired result.





$$
\begin{aligned}
\text{desired volume} &= 2\int_0^r 2\pi x\sqrt{r^2 - x^2}\,dx \\
&= 4\pi\int_0^r x\sqrt{r^2 - x^2}\,dx \\
&= \frac{4\pi}{(-2)}\int_0^r (-2)x\sqrt{r^2 - x^2}\,dx \\
&= -2\pi\int_{r^2}^0 u^{1/2}\,du \\
&= -2\pi \cdot \frac{2}{3}u^{3/2}\Big|_{r^2}^0 \\
&= -\frac{4\pi}{3}[0 - (r^2)^{3/2}] \\
&= -\frac{4\pi}{3}(-r^3) \\
&= \frac{4}{3}\pi r^3
\end{aligned}
$$

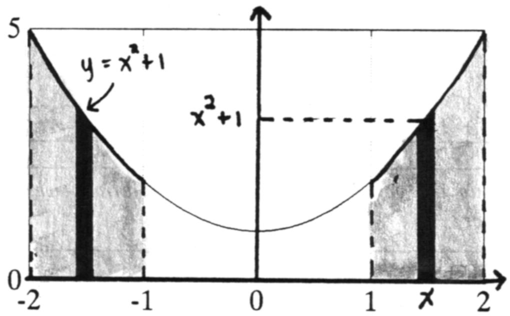
EXERCISE 2	♣ Cite a reason for every step in the previous example.

EXAMPLE

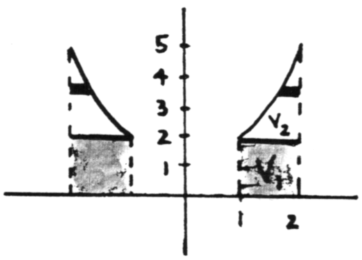
Problem: Revolve the region bounded by $y = x^2 + 1$, the x-axis, $x = 1$ and $x = 2$ about the y-axis. Find the volume of the resulting solid of revolution in *two ways*: using shells, and using disks. In each case, sketch the typical 'slice' (or 'slices'). Be sure to write complete mathematical sentences.

Solution using shells:

$$\int_1^2 2\pi x(x^2 + 1)\,dx = 2\pi\int_1^2 x^3 + x\,dx$$



$$
\begin{aligned}
&= 2\pi\left(\frac{x^4}{4} + \frac{x^2}{2}\right)\Big|_1^2 \\
&= 2\pi\left[(4+2) - \left(\frac{1}{4} + \frac{1}{2}\right)\right] \\
&= 2\pi\left(6 - \frac{3}{4}\right) = \frac{21\pi}{2}
\end{aligned}
$$



Solution using disks: This time, the solid must be separated into two pieces, because the 'slices' look different, depending upon the value chosen for y.

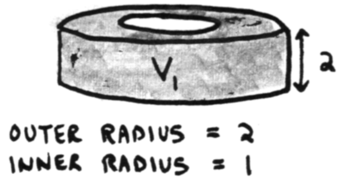
The volume V_1 of the bottom piece can be found without calculus; it is a cylinder, with a hole, of height 2. The outer radius is 2 and the inner radius is 1:

$$V_1 = \pi 2^2 \cdot 2 - \pi 1^2 \cdot 2 = 2\pi(2^2 - 1^2) = 2\pi(3) = 6\pi$$

The second volume V_2 requires using disks with holes:

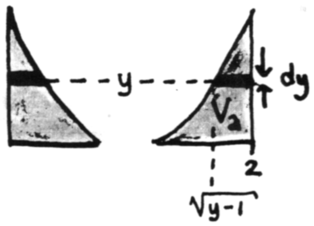
Let $y > 2$, and find the corresponding x-value:

$$y = x^2 + 1 \iff x^2 = y - 1 \iff x = \pm\sqrt{y-1}$$



Therefore, a typical slice for the upper section has inner radius $\sqrt{y-1}$, and outer radius 2, and thus has volume:

$$\pi 2^2 \cdot dy - \pi(\sqrt{y-1})^2 \cdot dy = \pi(4 - (y-1))\,dy = \pi(5-y)\,dy$$



'Summing' these disks as y travels from 2 to 5 yields:

$$\begin{aligned}\int_2^5 \pi(5-y)\,dy &= \pi(5y - \frac{y^2}{2})\Big|_2^5 \\ &= \pi\big[(25 - \frac{25}{2}) - (10-2)\big] \\ &= \pi(\frac{9}{2})\end{aligned}$$

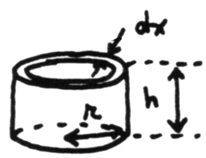
The total volume is

$$V_1 + V_2 = 6\pi + \frac{9}{2}\pi = \frac{21\pi}{2}\ ,$$

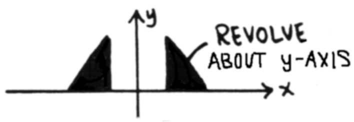
which agrees with the earlier result. You should be convinced that shells were *much easier* in this situation!

QUICK QUIZ
sample questions

1. What is the volume of the thin shell sketched here?



2. Revolve the area bounded by $y = x$ and the x-axis on $[0,2]$ around the y-axis. Use shells to find the volume of the resulting solid of revolution.
3. In the sketch below, would it be easier to use horizontal disks or shells to find the volume? Justify your answer.



KEYWORDS
for this section

The shell method for finding the volume of a solid of revolution; what is the volume of a typical thin slice?

END-OF-SECTION EXERCISES

Revolve each region described below about the y-axis. Find the volume of the resulting solid of revolution. Be sure to write complete mathematical sentences. Make a rough sketch of the solid under investigation.

1. Bounded by: $y = 2x$, $x = 0$, $x = 1$, and the x-axis
   (Find the volume in *two ways*; using shells, and using disks.)
2. Bounded by: $y = 2x$, $x = 1$, $x = 2$, and the x-axis
   (Find the volume in *two ways*; using shells, and using disks.)
3. Bounded by: $y = e^x$, $x = 0$, $x = 1$, and the x-axis
4. Bounded by: $y = e^x$, $x = 1$, $x = 2$, and the x-axis

Now, do these additional problems:

5. Derive the formula for the volume of a right circular cone of base radius r and height h, using shells.
6. Derive the formula for the volume of a cylinder of height h and base radius r, using shells.