MUST-KNOW MATERIAL FOR CALCULUS

MISCELLANEOUS:

interval notation: (a, b), $[a, b]$, $(a, b]$, (a, ∞), etc.

Rewrite radicals as fractional exponents: $\sqrt[3]{x} = x^{1/3}$, $\sqrt{x^3} = x^{3/2}$ etc.

An implication 'If A then B' is equivalent to its contrapositive 'If (not B) then (not A)'

To go from graph of $y = f(x)$ to graph of $x = f(y)$: take the (familiar) graph of $y = f(x)$, rotate 90 degrees clockwise, then flip about the horizontal axis.

TEST POINT METHOD: for solving $f(x) > 0$: there are only two types of places where a function can change from positive to negative (or vice versa): where it equals zero, or at a break. Locate all such places, and check the resulting subintervals.

GEOMETRY:

Circle with radius r: AREA $= \pi r^2$, CIRCUMFERENCE $= 2\pi r$, DIAMETER $= 2r$

Sphere with radius r: VOLUME $= \frac{4}{3}\pi r^3$, SURFACE AREA $= 4\pi r^2$

Area of a triangle:

base b and height h, AREA $= \frac{1}{2}bh$

sides a and b with included angle θ: AREA $= \frac{1}{2}ab\sin\theta$

Triangles: angles sum to 180°; longest side is opposite biggest angle, etc.

Consider an arbitrary triangle with angles A, B, C and opposite sides a, b, c:

Law of Sines: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Law of Cosines: $a^2 = b^2 + c^2 - 2bc\cos A$

Similar triangles: have the same angles; scaling factor in going from one to the other

Right triangles: have a 90° angle; longest side is called the hypotenuse;

the Pythagorean Theorem: $a^2 + b^2 = c^2$

Trapezoid with bases b_1 and b_2 and height h: AREA $= \frac{(b_1+b_2)}{2} \cdot h$ (average the bases and multiply by the height)

cylinder (2 parallel congruent plane figures of area A, perpendicular distance between planes is h): VOLUME $= Ah$

right circular cylinder: VOLUME $= \pi r^2 h$

cone (a plane figure of area A, a point, all lines connecting; h is perpendicular distance from point to plane): VOLUME $= \frac{1}{3}Ah$

right circular cone: VOLUME $= \frac{1}{3}\pi r^2 h$

TRIGONOMETRY:

RADIAN MEASURE: the radian measure of an angle is the length of the arc on the unit circle: positive is counterclockwise.

Right triangle definitions: SOHCAHTOA

Unit circle definitions; lay off angle x

$\sin x$ is the y-value of the point

$\cos x$ is the x-value of the point

$\tan x = \frac{\sin x}{\cos x}$

$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

$\sec x = \frac{1}{\cos x}$

$\csc x = \frac{1}{\sin x}$

$\sin^{-1} x = \arcsin x$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x.

$\cos^{-1} x = \arccos x$ is the angle between 0 and π whose cosine is x.

$\tan^{-1} x = \arctan x$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is x.

Note: $\sin^2 x$ means $\left(\sin x\right)^2$ etc.

Double-angle formulas: $\sin 2x = 2\sin x \cos x$; $\cos 2x = \cos^2 x - \sin^2 x$

the Pythagorean Identity: $\sin^2 x + \cos^2 x = 1$

Two special triangles: 30°-60°-90° and 45°-45°-90°

FUNCTIONS:

$f(x)$ is the output from the function f when the input is x.

Functions have the property that each input has exactly one corresponding output.

ZERO of a function: an input whose output is zero

DOMAIN of a function: set of allowable inputs

RANGE of a function: its output set

GRAPH of a function: the picture of its (input,output) pairs

GRAPHS of BASIC MODELS:

constant ($y = k$)

$y = x^2$ and higher powers

$y = x^3$ and higher powers

$y = \frac{1}{x}$

$y = \sqrt{x}$

$y = |x|$

$y = \mathrm{e}^x$

$y = \ln x$

$y = \sin x$

$y = \cos x$

$y = \tan x$

$y = \sec x$

$y = [[x]]$, the greatest integer function, $[[x]]$ is the greatest integer less than or equal to x

COMPOSITIONS OF FUNCTIONS: $f(g(x))$ means g acts first, f acts last

EVEN function: $f(-x) = f(x)$; when inputs are opposites, outputs are the same

ODD function: $f(-x) = -f(x)$; when inputs are opposites, outputs are opposites

ONE-TO-ONE FUNCTION: Each output has exactly one corresponding input; graph passes both a horizontal and vertical line test; the inputs and outputs can be tied together with strings

INVERSE FUNCTIONS:

If f is 1-1, then its inverse f^{-1} 'undoes' what f did: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

the domains and ranges of f and f^{-1} are switched

the graphs of f and f^{-1} are reflections about the line $y = x$

if (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}

TRANSFORMATIONS of functions: start with $y = f(x)$

working with y is intuitive:

$y = f(x) + 3$ moves up 3

$y = 3f(x)$ multiplies all y-values by 3 (vertical stretch)

$y = -f(x)$ multiplies the y-values by -1; reflects about the x-axis

working with x is counter-intuitive:

$y = f(x - 3)$; replace every x with $x - 3$; moves to the RIGHT 3

$y = f(3x)$; replace every x with $3x$; $(a, b) \mapsto (\frac{a}{3}, b)$; horizontal compression

$y = f(-x)$; replace every x with $-x$; reflects about the y-axis

LINES:

linear functions: $y = mx + b$ or $ax + by + c = 0$; equal changes in x give rise to equal changes in y

slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$; if $m = 3$, then the y-values are changing 3 times as fast as the x-values

point-slope form: $y - y_1 = m(x - x_1)$

parallel lines have the same slope; perpendicular lines have slopes that are opposite reciprocals

horizontal lines: $y = c$; have zero slope   vertical lines: $x = c$; have no slope

QUADRATIC FUNCTIONS:

$f(x) = ax^2 + bx + c$; graph as parabolas; $a > 0$ holds water, $a < 0$ sheds water

vertex: at $x = -\frac{b}{2a}$

POLYNOMIALS:

Let $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$.

degree of P: highest exponent

As $x \to \pm\infty$, a polynomial 'looks like' its highest power term.

The following are equivalent:

- c is a zero of P
- $x - c$ is a factor of $P(x)$
- $P(c) = 0$
- the point $(c, 0)$ is on the graph of P
- the graph of P crosses the x-axis at c
- $x - c$ goes into $P(x)$ evenly (remainder $= 0$)

EXPONENTIAL FUNCTIONS: $y = b^t$

Allowable bases: $b > 0$, $b \neq 1$

Increasing when $b > 1$; decreasing when $0 < b < 1$

Common form: $A(t) = A_0 \mathrm{e}^{kt}$; A_0 is the amount at time 0

Every exponential function can be written with ANY allowable base, so use whatever base is convenient.

For equal changes in x , y gets MULTIPLIED by a constant (that depends both on the base of the exponential function, and the change in x)

Doubling time: for an increasing exponential function, it always takes the same amount of time for a quantity to double

half-life: for a decreasing exponential function, it always takes the same amount of time for a quantity to be cut in half

LOGARITHMS: $y = \log_b x$

Allowable bases: $b > 0$, $b \neq 1$

Increasing when $b > 1$; decreasing when $0 < b < 1$

Laws work for all allowable bases:

$\ln x = \log_{\mathrm{e}} x$ is the natural logarithm

$\ln xy = \ln x + \ln y$ (the log of a product is the sum of the logs)

$\ln \frac{x}{y} = \ln x - \ln y$ (the log of a quotient is the difference of the logs)

$\ln x^y = y \ln x$ (you can bring powers down)

change-of-base formula: $\log_b x = \dfrac{\log_a x}{\log_a b}$

$y = e^x$ and $y = \ln x$ are inverse functions; use this idea to solve exponential and logarithmic equations

A log is an exponent! '$\log_3 5$' is the POWER that 3 must be raised to, to get 5

EXPONENTIAL FUNCTIONS grow faster than POWER FUNCTIONS grow faster than LOGARITHMIC FUNCTIONS

ABSOLUTE VALUE:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For $c > 0$,

$|x| < c \quad \iff \quad -c < x < c$

$|x| > c \quad \iff \quad x > c \text{ or } x < -c$

$|x| = c \quad \iff \quad x = \pm c$

$0 < |x - c| < \delta \quad \iff \quad x \in (c - \delta, c) \cup (c, c + \delta)$ (punctured neighborhood about c)

LIMITS:

Consider the limit statement: $\lim_{x \to c} f(x) = \ell$

low-level understanding: when x is close to c, $f(x)$ is close to ℓ

higher level: we can make the values of $f(x)$ as close to ℓ as we like, by taking x to be sufficiently close to c, but not equal to c

Precisely: $\forall\, \epsilon > 0\ \ \exists\, \delta > 0$ s.t. if $0 < |x - c| < \delta$ then $|f(x) - \ell| < \epsilon$

When we evaluate a limit as $x \to c$, we never let x equal c

$x \to c^+$ means x approaches c from the right-hand side

$x \to c^-$ means x approaches c from the left-hand side

LIMIT LAWS: Work nicely! Providing the individual limits exist, the limit of a sum is the sum of the limits (same for difference, products, quotients, etc.)

If you have a continuous function (see below) then evaluating a limit is as easy as DIRECT SUBSTITUTION.

BE CAREFUL! If you're working with a discontinuous function (e.g., greatest integer function, some piecewise-defined functions), then direct substitution MAY NOT WORK. Try l'Hospital's rule, renaming, graphing, etc.

An important limit: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

CONTINUITY:

Low-level understanding: no breaks in the graph

higher level: when inputs are close, outputs are close

The following are equivalent:

- f is continuous at c
- $\lim_{x \to c} f(x) = f(c)$

  (when f is continuous at c, then evaluating the limit is as easy as direct substitution)
- $\lim_{h \to 0} f(c+h) = f(c)$
- As $x \to c$, $f(x) \to f(c)$

INTERMEDIATE VALUE THEOREM:

Suppose f in continuous on $[a, b]$, and N is a number between $f(a)$ and $f(b)$. Then there exists a number c between a and b for which $f(c) = N$.

(If a graph has no breaks, and you travel along the graph from one point to another, you must pass through ALL the y-values in between; i.e., all the intermediate values.)

EXTREME VALUE (MAX/MIN) THEOREM:

Let f be continuous on a closed interval $[a, b]$. Then f attains both an absolute maximum value $f(c)$ and absolute minimum value $f(d)$ for some c and d in $[a, b]$.

(This theorem GUARANTEES the existence of a 'highest' and 'lowest' point on a graph under appropriate conditions.)

MEAN VALUE THEOREM:

Let f be differentiable on $[a, b]$. Then there exists a number c between a and b for which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

(This theorem guarantees a place where the instantaneous rate of change is the same as the average rate of change under appropriate conditions.)

DERIVATIVES:

The following are equivalent:

- $f'(c) = m$
- $\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h} = m$
- The slope of the tangent line to the graph of f at the point $(c, f(c))$ is m
- f is differentiable at c (and the value of the derivative is m)
- $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = m$
- The instantaneous rate of change of f at $(c, f(c))$ is m
- When $x = c$, the function values are changing m times as fast as the inputs

Suppose $f'(2) = 5$. Roughly: when x changes by 1 (from 2 to 3), we expect y to go up by ABOUT 5. Or, when x changes by -1 (from 2 to 1), we expect y to go down by ABOUT 5.

The UNITS of $f'(c)$ are the units of $f(x)$ (the outputs from f) divided by the units of x (the inputs to f)

If a function is differentiable, then its graph is SMOOTH: it has non-vertical tangent lines everywhere.

A function is NON-DIFFERENTIABLE at: vertical tangent lines; kinks; discontinuities

Theorem: If f is differentiable at x, then f is continuous at x

Contrapositive: If f is not continuous at x, then f is not differentiable at x

Leibniz notation versus prime notation: $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$ etc.

Linear Approximation (linearization): a function is best approximated at a point by its tangent line: at $(c, f(c))$ we have: $L(x) = f(c) + f'(c)(x - c) \approx f(x)$

If $f'(x) > 0$ then f is increasing

If $f'(x) < 0$ then f is decreasing

If the SLOPES are INCREASING (f' increasing; $f'' > 0$) then f is concave up

If the SLOPES are DECREASING (f' decreasing; $f'' < 0$) then f is concave down

Remember: a function can increase in basically three different ways: linearly, concave up; concave down

Inflection point: where the concavity changes (from concave up to down, or down to up): candidates are where $f''(c) = 0$ or $f''(c)$ does not exist.

LOCAL MAX/MIN:

A local max/min for a function can only occur at three types of places (called the 'critical points'):

- where $f'(c) = 0$
- where $f'(c)$ does not exist
- at ENDPOINTS of domain of f

So, to find max/min, locate all candidates, and check them.

Careful: a critical point is not necessarily a max or min!

FIRST DERIVATIVE TEST: Check signs of first derivative to the left/right of a candidate (where the function is continuous) to decide if it is a max or min.

Why is continuity needed? See the sketch below—the test would tell us that there's a local max at c!

SECOND DERIVATIVE TEST: If concave up at a candidate; it's a min. If concave down at a candidate, it's a max.

AVERAGE RATE OF CHANGE: The average rate of change of f on the interval $[a, b]$ is $\dfrac{f(b) - f(a)}{b - a}$; this is the slope of the line between $(a, f(a))$ and $(b, f(b))$

DIFFERENTIATION FORMULAS:

Be able to GENERALIZE all these formulas: replace x by $f(x)$, and multiply by $f'(x)$

$\frac{d}{dx} x^n = nx^{n-1}$ generalize: $\frac{d}{dx} \big(f(x)\big)^n = n\big(f(x)\big)^{n-1} \cdot f'(x)$

$\frac{d}{dx} cf(x) = c \frac{d}{dx} f(x)$ (you can slide constants out)

the derivative of a sum/difference is the sum/difference of the derivatives

$\frac{d}{dx} \mathrm{e}^x = \mathrm{e}^x$ (the y-value of the point tells you how fast the function is changing at that point)

PRODUCT RULE: $\frac{d}{dx} f(x)g(x) = f(x)g'(x) + g(x)f'(x)$ (the derivative of a product is NOT!! NOT!! NOT!! the product of the derivatives)

QUOTIENT RULE: $\dfrac{d}{dx} \dfrac{f(x)}{g(x)} = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$ (the derivative of a quotient is NOT!! NOT!! NOT!! the quotient of the derivatives)

x MUST BE MEASURED IN RADIANS FOR THESE FORMULAS TO BE CORRECT:

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin x$

$\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \cot x = -\csc^2 x$

$\frac{d}{dx} \sec x = \sec x \tan x$

$\frac{d}{dx} \csc x = -\csc x \cot x$

Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g(x) \cdot g'(x)$; how to differentiate composite functions

$\frac{d}{dx} a^x = a^x \ln a$

$$\frac{d}{dx} \arcsin x = \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

$\frac{d}{dx}\arccos x = \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\arctan x = \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

$\frac{d}{dx}\ln x = \frac{1}{x}$

$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$

derivative of an inverse function: $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

So if (a,b) is on the graph of f with slope of tangent line m,

then (b,a) is on the graph of f^{-1} with slope of tangent line $\frac{1}{m}$!

IMPLICIT DIFFERENTIATION:

Whenever you see y, treat it as a function of x and differentiate accordingly.

For example: $\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$

For example: $\frac{d}{dx}xy = x\frac{dy}{dx} + y$

LOGARITHMIC DIFFERENTIATION:

Use this to differentiate complicated products or quotients; also to differentiate variable stuff raised to a variable power. First take logs, then differentiate!

PARTICLE MOVING ON A NUMBER LINE:

Let $s(t)$ denote the position at time t.

Then, $s'(t) = v(t)$ is the velocity; positive is moving to the right; negative to the left.

$s''(t) = v'(t) = a(t)$ is the acceleration.

'Speeding up' means moving to the right faster and faster ($v(t) > 0$ and $a(t) > 0$) or moving to the left faster and faster ($v(t) < 0$ and $a(t) < 0$). Thus, the particle speeds up when velocity and acceleration have the same sign (both positive, or both negative).

Note: speed $= |v(t)|$

Suppose you're given the velocity of a particle traveling along a number line, $v(t)$. Then, total distance traveled from t_1 to t_2 is given by $\int_{t_1}^{t_2} |v(t)|\,dt$; i.e., integrate the speed.

However the total DISPLACEMENT is $\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$. Notice that if you start at 0, move to the right 5 and then to the left 5, your total displacement is 0 but the total distance traveled is 10.

RELATED RATE PROBLEMS:

Ask: What is changing with time? Rates are derivatives! Write down SOMETHING THAT IS TRUE that involves what you're interested in. (Look for: similar figures, right triangles, etc.)

Remember: if x is changing with time, then the derivative of x^2 is $2x\frac{dx}{dt}$.

OPTIMIZATION PROBLEMS:

Find the CANDIDATES for local max/min: endpoints, places where the derivative is zero, places where the derivative doesn't exist.

Use the First Derivative Test or Second Derivative Test to check whether they're a max or min.

If you want an ABSOLUTE max/min, find ALL the local max/min, and choose the highest/lowest from these.

Remember, you CAN'T USE YOUR CALCULATOR to locate max/mins; this is NOT an allowable operation!

ANTIDERIVATIVES:

A function $F(x)$ is an ANTIDERIVATIVE of $f(x)$ if and only if $F'(x) = f(x)$; i.e., F is a function whose derivative is f. Antiderivatives 'undo' derivatives. An antiderivative has a specified derivative, and this derivative determines the shape, but not the vertical translation. So, if you have ONE antiderivative, then you have an infinite number—they all differ by a constant.

The symbol $\int f(x)\,dx$ denotes all the antiderivatives of $f(x)$.

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ for $n \ne -1$

$\int \frac{1}{x} = \ln|x| + C$

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

$\int \mathrm{e}^{kx}\,dx = \frac{1}{k}\mathrm{e}^{kx} + C$

$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

Any CONTINUOUS function f has an antiderivative: the function $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$. That is, the function that finds AREA under the graph of f is an ANTIDERIVATIVE of f!

DEFINITE INTEGRALS:

the definite integral of f from a to b is denoted by $\int_a^b f(x)\,dx$ and is defined as follows:

Divide $[a,b]$ into n equal subintervals, each of length $\Delta x = \frac{b-a}{n}$. Choose x_i^* from the i^{th} subinterval. Then,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

You can approximate definite integrals with rectangles (left-hand; right-hand; midpoint), with trapezoids, even with parabolas.

The definite integral gives information about the (signed) area trapped between the graph of f and the x-axis: area above is treated as positive; area below is negative.

Caution: if $\int_a^b f(x)\,dx = 0$, this only means that there is the same amount of area ABOVE the x-axis as BELOW on the interval $[a, b]$.

When you have a definite integral problem that can be solved with simple geometry formulas (triangles, trapezoids, circles) then USE GEOMETRY to find the definite integral—it's much more efficient!

EVALUATION THEOREM: If F is any antiderivative of f, then $\int_a^b f(x)\,dx = F(b)-F(a)$.

TOTAL CHANGE THEOREM: When you integrate a rate of change, you get total change:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Rewrite this as

$$f(b) = f(a) + \int_a^b f'(x)\,dx$$

and think of it like this: If you want to know the value of f at b, first find the value of f at someplace you know (a), and then see how much f has CHANGED BY in going from a to b ($\int_a^b f'(x)\,dx$).

SUBSTITUTION METHOD for antidifferentiation/integration: Choose u to be something whose derivative is in the integrand, perhaps off by a constant. Often, u is something in parentheses, the argument of a function, something in an exponent, etc. Be sure to change the limits if you have a definite integral.

APPLICATIONS OF INTEGRATION:

AREAS BETWEEN CURVES: Find the intersection points, write the area of a typical 'slice' and 'sum' appropriately.

vertical slices: AREA $= \int_a^b \big(f(x) - g(x)\big)\,dx$

horizontal slices: AREA $= \int_c^d \big(f(y) - g(y)\big)\,dy$ (Will need to solve for x in terms of y)

VOLUMES OF REVOLUTION:

DISK METHOD: revolve $y = f(x)$ about the x-axis on $[a, b]$; volume of the resulting solid is $\int_a^b \pi\big(f(x)\big)^2\,dx$.

SHELL METHOD: revolve $y = f(x)$ on $[a, b]$ about the y-axis; volume of the resulting solid is $\int_a^b 2\pi x f(x)\,dx$.

AVERAGE VALUE OF A FUNCTION: the average value of $f(x)$ on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$; you're 'summing up' the outputs from a to b (the integral), and then 'dividing by how many you have' (the length of the interval). If you 'smush' the area into rectangular shape, the average value gives the height of the rectangle.

Caution: Don't mix up 'average rate of change' and 'average value'!

CONNECTION BETWEEN **average value** and the **average rate of change**:

$\dfrac{1}{b-a}\displaystyle\int_a^b f'(x)\,dx = \dfrac{f(b) - f(a)}{b-a}$: 'averaging' the values of $f'(x)$ on $[a, b]$ gives the average rate of change of f on $[a, b]$

SEPARABLE DIFFERENTIAL EQUATIONS:

Get all the y's on one side, and all the x's on the other side. Integrate. Don't forget the constant of integration. Use a given condition to solve for this constant.

SLOPE FIELDS:

Slope fields help us to visualize the solutions to first-order differential equations. Get a formula for the derivative in terms of x and y; find the slope at many different points. The resulting 'field of slopes' helps us to see the shapes of the solution curves.