11. INTRODUCTION TO FRACTIONS

What is a fraction?

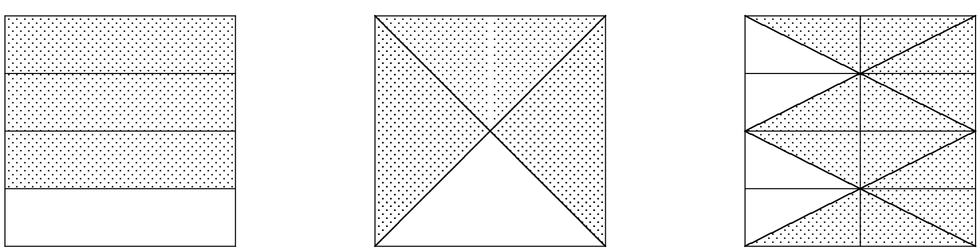
A fraction is a mathematical expression of the form $\frac{N}{D}$, where N is called the *numerator* and D is called the *denominator*. Even though $\frac{1}{2}$ and 0.5 are the same number, $\frac{1}{2}$ is called a fraction, but 0.5 is not. (What is 0.5 called?) Even though you sometimes see fractions written with a diagonal division symbol (like 1/2) instead of a horizontal fraction bar (like $\frac{1}{2}$), the latter representation is strongly preferred in algebra and beyond, for reasons to be discussed later. This section and the next will review the fraction concept and the basic skills that you need for working with fractions. The ideas are presented rather quickly, because you've very likely worked with fractions extensively in earlier math courses.

EXERCISES

1. Which of the following expressions are fractions? In any fraction, identify both the numerator and the denominator.
   a. $\dfrac{1}{4}$
   b. 0.25
   c. $\dfrac{x}{1}$
   d. x
   e. $\dfrac{1+x}{x-2}$
   f. 5%
   g. $\dfrac{5}{100}$
2. Rename each of the following expressions as a fraction. (There are many possible correct answers.)
   a. 0.03
   b. 7%
   c. y
   d. $1+3$

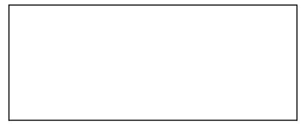
fraction concept: part of a whole

One important use of fractions is to represent part of a whole. To illustrate, suppose you want $\frac{3}{4}$ (three-fourths) of an object. Start by dividing the object into 4 equal-size pieces, and then take 3 of the pieces. Notice that the denominator tells how many equal pieces the object is to be divided into, and the numerator tells how many of these to take. Each of the following sketches has three-fourths of the object shaded:



EXERCISES

3. Sketch two-fifths of the rectangle below, in three different ways.



locating fractions on a number line

It's important that you can locate fractions like $\frac{2}{7}$, $\frac{14}{3}$, and $-\frac{1}{4}$ on a number line. This makes use of the 'part of a whole' concept, as discussed next.

when the numerator is less than the denominator, the number is between 0 and 1

In a fraction $\frac{N}{D}$, if the numerator N is less than the denominator D, then the fraction is between 0 and 1. For example, to locate $\frac{2}{7}$, divide the distance between 0 and 1 into 7 equal segments, and then take 2 of these:

EXERCISES

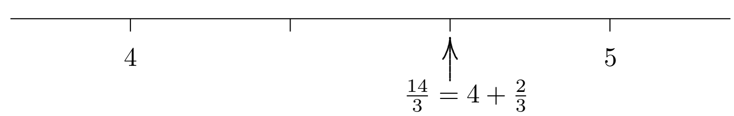
4. Locate each of the following fractions on a number line: $\dfrac{2}{5}$, $\dfrac{4}{10}$, $\dfrac{2}{3}$, and $\dfrac{3}{8}$.

when the numerator is greater than the denominator, rewrite first

If the numerator is greater than the denominator, rewrite first, to get a whole number part and a fraction part. For example, consider the fraction $\frac{14}{3}$. How many times does 3 go into 14? Answer: 4. How many are left over? Answer: 2. Thus, $\frac{14}{3} = 4 + \frac{2}{3}$. Alternately, you can write:

$$\frac{14}{3} = \frac{12+2}{3} = \frac{12}{3} + \frac{2}{3} = 4 + \frac{2}{3}\,.$$

Thus, go to the right 4, and then to the right $\frac{2}{3}$ more:

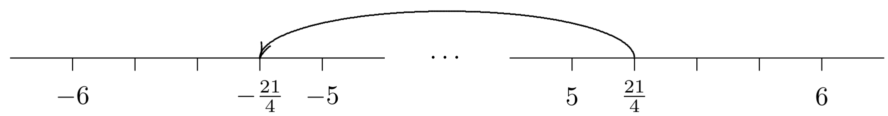


locating negative numbers

To locate a negative number, just locate the corresponding positive number, and then take its opposite. For example, to locate $-\frac{21}{4}$, first work with the positive number $\frac{21}{4}$:

$$\frac{21}{4} = \frac{20+1}{4} = \frac{20}{4} + \frac{1}{4} = 5 + \frac{1}{4} .$$

Then, to locate $-\frac{21}{4}$, just go to the opposite of $\frac{21}{4}$:



EXERCISES

5. Locate each of the following fractions on a number line: $\dfrac{27}{4}$, $\dfrac{10}{7}$, and $-\dfrac{38}{5}$.

$\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$

Observe the property of fractions that has been used in the previous argument: For all real numbers A and B, and for $C \neq 0$,

$$\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$$

That is, when you have a sum in the numerator, you can split the fraction into two pieces, each having the same denominator as the original fraction.

every subtraction problem is an addition problem in disguise

The property just mentioned also works for subtraction, because every subtraction problem is an addition problem in disguise: $A - B = A + (-B)$. That is, to subtract a number is the same as adding its opposite. Indeed, when mathematicians say 'sum,' they realize that they're covering both addition *and* subtraction. Thus, mathematicians would call the expression $x - y + z$ a sum, even though there is both addition and subtraction being used. Thus,

$$\frac{A-B}{C} = \frac{A+(-B)}{C} = \frac{A}{C} + \frac{-B}{C} = \frac{A}{C} - \frac{B}{C} .$$

Caution!!

CAUTION! You cannot split a fraction if the sum is in the denominator. Clearly, $\frac{2}{1+1} \neq \frac{2}{1} + \frac{2}{1}$.

EXERCISES

6. Break each fraction into two or more simpler fractions, if possible:
   a. $\dfrac{x+1}{y}$
   b. $\dfrac{x}{y+1}$
   c. $\dfrac{A+B}{2}$
   d. $\dfrac{x+y-z}{D}$
   e. $\dfrac{x+y-z}{D+2}$
   f. $\dfrac{x}{D+2}$

fractions involving zero:

0 *in the numerator; fraction equals* 0

Any fraction with 0 in the numerator and a nonzero number in the denominator equals 0. That is,

$$\frac{0}{N} = 0 \quad \text{for all } N \neq 0\,.$$

Thus, the number 0 has many different fraction names:

$$0 = \frac{0}{3} = \frac{0}{1.5} = \frac{0}{-0.09} = \dots$$

0 *in the denominator: fraction is not defined; division by* 0 *is not allowed*

However, any fraction with 0 in the denominator is undefined—does not exist—is not a real number. This is often stated by saying that *division by* 0 *is not allowed*. So when you come across something like $\frac{3}{0}$, you can say any of the following:

- $\frac{3}{0}$ does not exist
- $\frac{3}{0}$ is not defined
- division by zero is not allowed

EXERCISES

7. Give a simpler name for each of the following fractions involving 0, if possible.
   a. $\dfrac{0}{6}$
   b. $\dfrac{6}{0}$
   c. $\dfrac{0}{0}$
   d. $\dfrac{-1+1}{5.79}$
   e. $\dfrac{7}{3-3}$
   f. $\dfrac{2-2}{N}$, where $N \neq 0$

fractions involving zero: the reason why

Every fraction $\frac{N}{D}$ can be re-written as $N \cdot \frac{1}{D}$. For example, $\frac{3}{4} = 3 \cdot \frac{1}{4}$. Thus, $\frac{0}{3} = 0 \cdot \frac{1}{3} = 0$. Thus, it's easy to see that fractions with a numerator of zero must equal zero.

The reason that division by zero is not defined is a bit more subtle, and this argument is presented as ★ material next. Remember that ★ material is for more advanced readers, and can be skipped without any loss of continuity.

★
behavior of fractions involving 0

Let $\frac{a}{b} = c$, where c is the number that makes the equivalent equation $a = bc$ true.

If $a = 0$ and $b \neq 0$, then c must equal 0.

If $b = 0$ and $a \neq 0$, then $a = bc$ is false for all c, so division by zero is not defined.

If $a = 0$ and $b = 0$, then $a = bc$ is true for all real numbers c, so $\dfrac{a}{b}$ would not be unique; thus $\frac{0}{0}$ is not defined.

EXERCISES
web practice

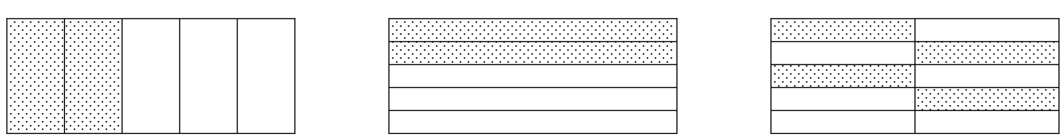
Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTIONS TO EXERCISES: INTRODUCTION TO FRACTIONS

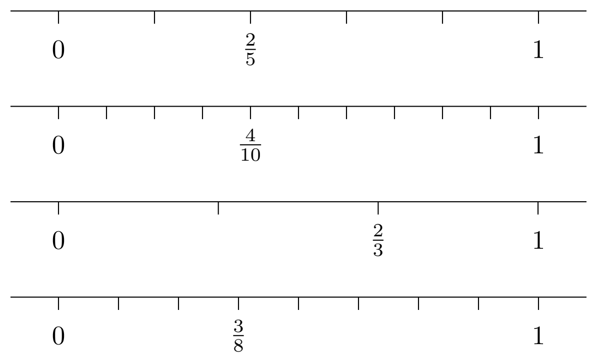
1. a. fraction; numerator is 1, denominator is 4
b. not a fraction
c. fraction; numerator is x, denominator is 1
d. not a fraction
e. fraction; numerator is $1 + x$, denominator is $x - 2$
f. not a fraction
g. fraction; numerator is 5, denominator is 100

2. a. $0.03 = \frac{3}{100}$
b. $7\% = \frac{7}{100}$
c. $y = \frac{y}{1}$
d. $1 + 3 = \frac{4}{1} = \frac{8}{2}$

3. There are many correct solutions. Here are three of the more obvious ones:

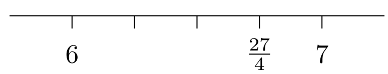


4.

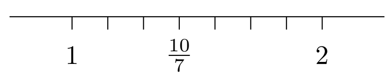


5.

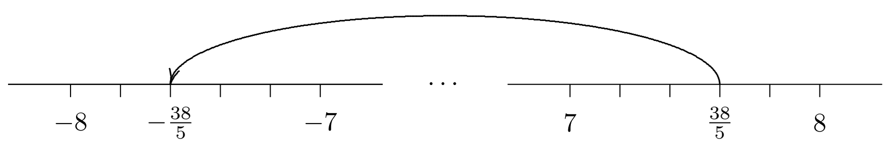
$$\frac{27}{4} = 6 + \frac{3}{4}$$



$$\frac{10}{7} = 1 + \frac{3}{7}$$



$$\frac{38}{5} = 7 + \frac{3}{5}$$



6. a. $\dfrac{x+1}{y} = \dfrac{x}{y} + \dfrac{1}{y}$

b. $\dfrac{x}{y+1}$ cannot be simplified

c. $\dfrac{A+B}{2} = \dfrac{A}{2} + \dfrac{B}{2}$

d. $\dfrac{x+y-z}{D} = \dfrac{x}{D} + \dfrac{y}{D} - \dfrac{z}{D}$

e. $\dfrac{x+y-z}{D+2} = \dfrac{x}{D+2} + \dfrac{y}{D+2} - \dfrac{z}{D+2}$

f. $\dfrac{x}{D+2}$ cannot be simplified

7. a. $\dfrac{0}{6} = 0$

b. $\dfrac{6}{0}$ does not exist

c. $\dfrac{0}{0}$ does not exist

d. $\dfrac{-1+1}{5.79} = 0$

e. $\dfrac{7}{3-3}$ does not exist

f. $\dfrac{2-2}{N} = 0$, when $N \neq 0$