15. NUMBERS HAVE LOTS OF DIFFERENT NAMES!

a fun type of game with numbers

There are lots of number games that can make you look clairvoyant. One such game goes something like this:

- YOU (speaking to another person): Think of a number, but *don't tell me what it is*! Do (*this and this and this*) to the number. I'll bet you ended up with (*some number*)—am I right?
- OTHER PERSON: You're right! How did you do that?

one such game

One such game is described below. A couple examples of 'playing the game' are given after the instructions.

Get yourself a piece of paper and a pencil, and follow the instructions as you read through these steps. Use a calculator if needed.

- STEP 1: Take the number of pets you own (0, 1, 2, etc.), and add 2 to this number. Write down your result, and circle it. If you own fewer than two pets, go on to STEP 2. Otherwise, skip to STEP 3.
- STEP 2: (Only do this step if you own fewer than 2 pets.) Subtract the number of pets you own from 2. (That is, go '$2 -$ *number of pets*'.) Multiply the result by your circled number. Write down the result, and put a box around it. Skip to STEP 4.
- STEP 3: (Only do this step if you own 2 or more pets.) Take your number of pets, and subtract 2. (That is, go '*number of pets* $- 2$'.) Take the opposite of your result. Multiply by your circled number. Write this new number down, and put a box around it. Go to STEP 4.
- STEP 4: Take the number of pets you own, multiply it by itself, and add this result to the boxed number.

Providing the instructions are given and followed correctly, you'll *always* end up with the number 4!

playing the game:

Here are a couple examples of playing the game.

3 pets

First, suppose you own 3 pets (use a calculator as needed):

- STEP 1: $\overbrace{3}^{\text{3 pets}} + 2 = 5$; write down the number 5 , and circle it.
  Since you own more than 2 pets, go to STEP 3.
- STEP 3: $\overbrace{3}^{\text{3 pets}} - 2 = 1$; opposite is -1 ; $(-1)\cdot 5 = \boxed{-5}$

  Write down the number $\boxed{-5}$ and put a box around it.
- STEP 4: $\overbrace{3}^{\text{3 pets}} \cdot \overbrace{3}^{\text{3 pets}} = 9$; $9 + (\ \boxed{-5}\ ) = 4$

1 pet

Now, suppose you own 1 pet:

- STEP 1: $\overbrace{1}^{\text{1 pet}} + 2 = 3$; write down the number 3, and circle it.
  Since you own fewer than 2 pets, go to STEP 2.
- STEP 2: $2 - \overbrace{1}^{\text{1 pet}} = 1$; $1 \cdot 3 = \boxed{3}$; write down the number $\boxed{3}$, and put a box around it. Go to STEP 4.
- STEP 4: $\overbrace{1}^{\text{1 pet}} \cdot \overbrace{1}^{\text{1 pet}} = 1$; $1 + \boxed{3} = 4$

game variations

You can vary the game as much as you'd like by replacing the 'number of pets' with anything you might not know about the other person:

- the person's age
- how many cavities the person has
- the number of cookies the person ate yesterday
- how many times the person exercised last week

... and on and on and on.

Why does this work?

Why does this work? It's a consequence of an extremely important aspect of mathematics: *numbers have lots of different names*. All the author did in constructing this particular little 'game' was to come up with two slightly unusual names for the number 4. (These names are given below: however, you may not yet have the mathematical tools needed to recognize them as the number 4.)

★
the names for 4 that were used in the previous game

If $x \geq 2$, here's the name for 4 that was used:

$$4 = -(x-2)(2+x) + x^2$$

If $x < 2$, here's the name for 4 that was used:

$$4 = (2-x)(2+x) + x^2$$

EXERCISES

Using a calculator, if needed:

1. Play the game, beginning with the number 0.
2. Play the game, beginning with the number 7.

English synonyms

In English, words that look different, but have (nearly) the same meaning, are called *synonyms*. For example, 'anxious' and 'fretful' are synonyms. But, *there is no language in the world where the idea of 'different name, same meaning' is more prevalent than in the language of mathematics.*

Different names can reveal various properties that a number has. For example, suppose that you have 36 pieces of candy. Here's the type of information that four different names for 36 might reveal:

name for 36	information revealed by name
$3 \cdot 12$	36 pieces of candy can be evenly distributed among 3 kids, by giving 12 pieces to each
$2 \cdot 8 + 5 \cdot 4$	give 8 pieces to each of 2 kids, and 4 pieces to each of 5 kids
$5 \cdot 7 + 1$	give 7 pieces to each of 5 kids, with 1 piece left over
$(72)(\frac{1}{2})$	give half a piece to each of 72 kids

EXERCISES

3. The *same name* for a number can sometimes reveal different information, depending upon its interpretation. For example, the name '$36 = 3 \cdot 12$' might be interpreted as: '36 pieces of candy can be evenly distributed among 12 kids, by giving 3 pieces to each'. (Compare this with the previous interpretation.)

Give interpretations, different from those in the previous table, for the names:

(a) $2 \cdot 8 + 5 \cdot 4$

(b) $5 \cdot 7 + 1$

4. Fill in the blanks below, by providing either the appropriate name for the number 60 (thought of as 60 pieces of candy), or the information revealed by the given name:

name for 60	information revealed by name
$6 \cdot 10$	
	60 pieces of candy can be evenly distributed among 3 kids, by giving 20 pieces to each
	give 7 pieces of candy to each of 8 kids, with 4 pieces left over
$16 \cdot 3 + 2 \cdot 6$	
$\frac{1}{3}(180)$	

getting a name that is useful to you

The ability to take a number, and *get a name for that number that is useful to you*, is a key to success in mathematics. There are two favorite ways to get a new *name* for a number (without changing where the number lives on a number line):

- by adding zero; or
- by multiplying by one.

Indeed, the numbers zero (0) and one (1) have *very special properties* in our number system: look below to see how you might be told about these properties, using the language of mathematics. Afterwards, the word 'theorem' (THEE-rum) is discussed; and then the *power* that these seemingly trivial properties give us is investigated.

THEOREM

special properties of 0 and 1

For all real numbers x ,

$$x + 0 = x \quad \text{and} \quad x \cdot 1 = x \ .$$

For all nonzero real numbers x ,

$$\frac{x}{x} = x \cdot \frac{1}{x} = 1 \ .$$

What is a 'Theorem'?

A *theorem* is the name that mathematicians give to something having two properties:

- it is *true*; and
- it is *important*.

★ The author just told a little white lie. Actually, a theorem is a true, important, statement that has been *proved*.

proving a result

Fortunately, mathematicians have very careful ways of verifying that a result is *true*. The process of showing that a result is *true* is called **proving the result**. (A non-mathematician once asked a mathematician: "What do you do?" The mathematician's answer? "I prove theorems.")

However, people don't always agree about how *important* something is. Mathematics is no exception. Things that don't seem quite worthy of being called 'theorems' are given other names:

- A **proposition** (prop–a–ZI–shun) is not quite important enough to be called a theorem.
- A **lemma** (LEM–ma) is usually a stepping-stone to a theorem.
- A **corollary** (KORE–a–larry) is usually an interesting consequence of a theorem.

theorems are to a mathematician, as tools are to a carpenter

Theorems are to a mathematician, as tools are to a carpenter. With the correct use of appropriate tools, a carpenter can build a beautiful, structurally sound building. With the correct use of appropriate theorems, mathematicians can give beautiful, structurally sound solutions to a wide variety of problems.

translating the previous theorem

Next, let's discuss the content of the previous theorem. The following exercises lead you through the translation of the first part:

EXERCISES

(Refer to the previous theorem for all these exercises.)

5. What is the universal set for x in the sentence '$x + 0 = x$'? How do you know?

6. Translate: 'For all real numbers x, $x + 0 = x$.' That is, what is this 'for all' sentence telling you that you can DO?

7. What is the universal set for x in the sentence '$x \cdot 1 = x$'? How do you know?

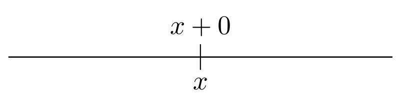
8. Translate: 'For all real numbers x, $x \cdot 1 = x$.' That is, what is this 'for all' sentence telling you that you can DO?

names for 0

The sentence

$$\text{'For all real numbers } x\,,\ \ x + 0 = x\text{'}$$

informs us that adding zero does not change where a number lives; it only provides a new *name* for the number. That is, no matter what number x is currently 'holding', x and $x+0$ live at exactly the same place on a real number line:



zero has lots of different names!

When we rename a number by adding zero, we usually *don't use the name '0' for zero*. (Like all other numbers, zero has lots of different names!) So, what name(s) for zero are usually used? Since a number, when added to its opposite, always yields zero, we have the *ability* to get *lots of different names for zero*:

- Want to bring 2 into the picture? Then you might choose to add 0 in any of these forms:

  $$2 + (-2) \qquad \text{or} \qquad (-2) + 2 \qquad \text{or, most simply,} \qquad 2 - 2\,.$$

- Want to bring $\frac{1}{3}$ into the picture? Then you might choose to add 0 in any of these forms:

  $$\tfrac{1}{3} + (-\tfrac{1}{3}) \qquad \text{or} \qquad (-\tfrac{1}{3}) + \tfrac{1}{3} \qquad \text{or, most simply,} \qquad \tfrac{1}{3} - \tfrac{1}{3}\,.$$

EXERCISE

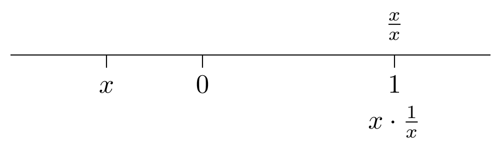
9. Get names for zero that use each of the following numbers:
   (a) 5
   (b) $\frac{1}{2}$
   (c) 3.2
   (d) -7

names for 1

Similarly, when we rename a number by multiplying by one, we usually *don't use the name '* 1 *' for one.* (Like all other numbers, ' 1 ' has lots of different names!) Indeed, the second part of the previous theorem provides us with a multitude of names for the number 1:

$$\text{For all nonzero real numbers } x\,,\ \ \frac{x}{x} = x \cdot \frac{1}{x} = 1\,.$$

Translation: As long as x isn't zero, then ' $\frac{x}{x}$ ' and ' $x \cdot \frac{1}{x}$ ' are both just different names for the number 1:



Why are ' $\frac{x}{x}$ *' and '* $x \cdot \frac{1}{x}$ *' names for '* 1 *'?*

The name ' $\frac{x}{x}$ ' for ' 1 ' (which is the 'horizontal fraction' form of $x \div x$) is a consequence of the fact that any nonzero number, when divided by itself, gives 1. For example,

$$\frac{5}{5} = 5 \div 5 = 1 \ \text{ and } \ \frac{1/2}{1/2} = \frac{1}{2} \div \frac{1}{2} = 1 \ \text{ and } \ \frac{-1.35}{-1.35} = -1.35 \div (-1.35) = 1\ .$$

everything can be done with multiplication

Now, how about the name ' $x \cdot \frac{1}{x}$ ' for ' 1 '? Recall from earlier sections that division is superfluous—it's not needed. Everything can be done with multiplication alone.

the reciprocal of a nonzero real number

Remember that the **reciprocal** (re-SI-pro-kul) of a nonzero number x is the number $\frac{1}{x}$.

For example, the reciprocal of 2 is $\frac{1}{2}$, and the reciprocal of 1.3 is $\frac{1}{1.3}$.

Then, *dividing by* x is the same as *multiplying by the reciprocal of* x. That is,

$$x \ \overbrace{\div \ \ x}^{\text{dividing by } x} \ \ \overbrace{=}^{\text{is the same as}} \ \ x \ \ \overbrace{\cdot \ \ \frac{1}{x}}^{\text{multiplying by the reciprocal of } x}$$

' 1 ' has lots of different names!

Now you have the *ability* to get *lots of different names for* ' 1 ':

- Want to bring 2 into the picture? Then you might choose to multiply by 1 in any of these forms:
$$\frac{2}{2} \qquad \text{or} \qquad 2 \cdot \frac{1}{2} \qquad \text{or} \qquad \frac{1}{2} \cdot 2$$
- Want to bring $\frac{1}{3}$ into the picture? Then you might choose to multiply by 1 in any of these forms:
$$\frac{1/3}{1/3} \qquad \text{or} \qquad 3 \cdot \frac{1}{3} \qquad \text{or} \qquad \frac{1}{3} \cdot 3$$

EXERCISE

10. (Compare with exercise 9.) Get names for '1' that use each of the following numbers:

(a) 5

(b) $\frac{1}{2}$

(c) 3.2

(d) -7

sentences of the form $a = b = c$

A common mathematical shorthand which deserves some attention was introduced in the previous theorem. The sentence

$$\frac{x}{x} = x \cdot \frac{1}{x} = 1$$

has the form

$$a = b = c\ ;$$

that is,

$$\textit{something} = \textit{something} = \textit{something}\ .$$

When people write '$a = b = c$', they *really* mean to write

$$a = b \ \text{ and } \ b = c\ ,$$

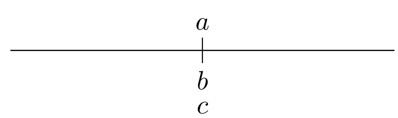
but they get lazy. That is,

'$a = b = c$' is a shorthand for '$a = b$ and $b = c$'.

When is a sentence of the form $a = b = c$ *true?*

In order for a sentence of the form '$a = b = c$' to be true, BOTH '$a = b$' and '$b = c$' must be true. That is, a must equal b, and b must equal c. It follows that a must equal c.

Consequently, when a sentence of the form '$a = b = c$' is TRUE, this means that a, b, and c are just different names for the same number:



★ *the mathematical sentence '$a = b = c$'*

Precisely: For all real numbers a, b, and c,

$$a = b = c \iff \big(a = b \text{ and } b = c\big)\ .$$

The mathematical word 'AND' is defined via the following truth table:

A	B	A AND B
T	T	T
T	F	F
F	T	F
F	F	F

Thus, an 'AND' sentence is true only when both subsentences are true.

The mathematical words 'AND' and 'OR', and the symbol $\iff$, will be discussed in future sections.

EXERCISES

11. Decide whether each sentence is true, false, or sometimes true/sometimes false:

(a) $1 = \frac{4}{4} = 4 \cdot \frac{1}{4}$

(b) $2 + 3 = 5 + 1 = 6$

(c) $1 + 2 + 3 = 1 + 5 = 6$

(d) $1 = \frac{t}{t} = t \cdot \frac{1}{t} = \frac{1}{t} \cdot t$

(e) $4 = 4 + 0 = 0 + 4$

(f) $1 + (2 + 3) + 4 = 5 = 1 + 5 = 6 + 4 = 10$

12. The sentence '$a = b = c = d$' is a shorthand—for what?

Why isn't '$\frac{0}{0}$' a name for the number '1'?

Recall that 0 was excluded from the universal set for x in the sentence

$$\frac{x}{x} = x \cdot \frac{1}{x} = 1 \ .$$

Why is this? That is, why isn't '$\frac{0}{0}$' a name for the number '1'? Here's the idea. We want the pair of sentences

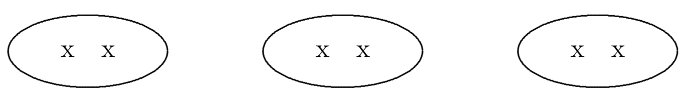
$$\frac{a}{b} = c \qquad \text{and} \qquad a = b \cdot c$$

to *always* have the same truth values. If one is true, so is the other. If one is false, so is the other. Study the examples below:

$$\text{compare} \quad \overbrace{\frac{6}{3} = 2}^{\text{true}} \quad \text{and} \quad \overbrace{6 = 3 \cdot 2}^{\text{true}}$$

$$\text{compare} \quad \overbrace{\frac{9}{5} = 3}^{\text{false}} \quad \text{and} \quad \overbrace{9 = 5 \cdot 3}^{\text{false}}$$

The idea is illustrated by the pair of (true) sentences '$\frac{6}{3} = 2$' and '$6 = 3 \cdot 2$':



$\frac{6}{3} = 2$	Take 6 objects and divide them into 3 equal piles by putting 2 in each pile.
$6 = 3 \cdot 2$	Put the piles back together: 3 piles, with 2 in each, gives 6 objects.

suppose that ' $\frac{0}{0}$ ' is to be a name for the number c

Keeping this in mind, let's claim that ' $\frac{0}{0}$ ' is supposed to be a name for the number ' c '. Then, the following two equations would need to have the same truth values:

$$\frac{0}{0} = c \qquad \text{and} \qquad 0 = 0 \cdot c$$

Notice, however, that the equation ' $0 = 0 \cdot c$ ' is true for *all* real numbers c! (Why? Any number, when multiplied by zero, gives zero.) Therefore, since the equations are supposed to have the same truth values, ' $\frac{0}{0} = c$ ' would also need to *always* be true.

But if ' $\frac{0}{0} = c$ ' is always true, then the symbol ' $\frac{0}{0}$ ' would have to be a name for *every possible real number*. Think of the mass confusion that would result!

In one instance, ' $\frac{0}{0}$ ' might represent the number 1. In another instance, it might represent the number -5. To avoid the problem entirely, it has been decided that ' $\frac{0}{0}$ ' is *undefined*—it doesn't represent *any* real number—it's nonsensical—it's not allowed. Sorry!

division by zero is not allowed

So, ' $\frac{0}{0}$ ' is undefined. Similarly, the symbol ' $\frac{a}{0}$ ' is undefined for all nonzero real numbers a. To see why, consider something like ' $\frac{2}{0}$ '. What number should ' $\frac{2}{0}$ ' represent? Let's claim that ' $\frac{2}{0}$ ' is a name for the number ' c '. Then, the following two equations must have the same truth values:

$$\frac{2}{0} = c \qquad \text{and} \qquad 2 = 0 \cdot c$$

Notice, however, that the equation ' $2 = 0 \cdot c$ ' is false for *all* real numbers c! (Why? Any number, when multiplied by zero, gives zero; and zero is not equal to 2.) Therefore, ' $\frac{2}{0} = c$ ' is also *always* false. Consequently, the symbol ' $\frac{2}{0}$ ' doesn't represent any real number.

This discussion is usually summarized by saying that 'division by zero isn't allowed'.

the power of adding 0 and multiplying by 1

The seemingly trivial properties:

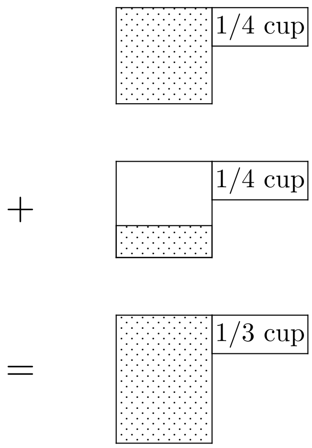
- adding 0 to a number doesn't change it
- multiplying a number by 1 doesn't change it

become *incredibly powerful tools*, when used in conjunction with other arithmetic skills. They give you the power to *get a name for a number that is useful for you*. Here's an example:

making cornbread

The author has a favorite cornbread recipe: easy and quick to make, healthful, inexpensive, and delicious. The only problem is that making it required dirtying a 1-cup measure, a $\frac{1}{4}$-cup measure, and a $\frac{1}{3}$-cup measure. Why wash three utensils when only one would suffice? So, the author decided to 'rename' the ingredient amounts, so that only a $\frac{1}{4}$-cup measure is needed.

For example, the recipe calls for $\frac{1}{3}$ cup vegetable oil. A new name for ' $\frac{1}{3}$ ' is needed, that expresses $\frac{1}{3}$ in terms of $\frac{1}{4}$. The re-naming process illustrated next exploits certain arithmetic skills: some re-grouping, re-ordering, and working with fractions and signed $(+/-)$ numbers. For now, don't worry if there are some steps that you don't fully understand; concentrate mainly on the appearance of 'adding zero' and 'multiplying by one':



$$\frac{1}{3} = \frac{1}{3} + \overbrace{(\frac{1}{4} - \frac{1}{4})}^{\text{add zero}}$$

Want to bring $\frac{1}{4}$ into the picture? Add zero in an appropriate form!

$$= \frac{1}{4} + (\frac{1}{3} - \frac{1}{4})$$

re-order; re-group

$$= \frac{1}{4} + \frac{1}{12}$$

arithmetic with fractions: $\frac{1}{3} - \frac{1}{4} = \frac{1}{3}\cdot\frac{4}{4} - \frac{1}{4}\cdot\frac{3}{3} = \frac{4}{12} - \frac{3}{12} = \frac{1}{12}$; thus, $\frac{1}{3} - \frac{1}{4}$ goes by the name $\frac{1}{12}$

$$= \frac{1}{4} + \frac{1}{12} \cdot \overbrace{(\ \frac{1}{4} \cdot 4\ )}^{\text{multiply by 1}}$$

Want to bring $\frac{1}{4}$ into the picture again? Multiply by 1 in an appropriate form!

$$= \frac{1}{4} + (\frac{1}{12} \cdot 4) \cdot \frac{1}{4}$$

re-order; re-group

$$= \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4}$$

arithmetic with fractions: $\frac{1}{12} \cdot 4$ goes by the simpler name $\frac{1}{3}$

The final name for $\frac{1}{3}$ is much more useful: put in $\frac{1}{4}$ cup, plus one-third of a $\frac{1}{4}$ cup!

one more time, with feeling

The author could have gotten the 'new name' by applying the 'transforming tools' in a different way:

$$\frac{1}{3} = \frac{1}{3} \cdot \overbrace{(\ \frac{1}{4} \cdot 4\ )}^{\text{multiply by 1}}$$

Want to bring $\frac{1}{4}$ into the picture? Multiply by 1 in an appropriate form!

$$= (4 \cdot \frac{1}{3}) \cdot \frac{1}{4}$$

re-order; re-group

$$= \frac{4}{3} \cdot \frac{1}{4}$$

arithmetic with fractions: $4 \cdot \frac{1}{3}$ goes by the name $\frac{4}{3}$

$$= (1 + \frac{1}{3}) \cdot \frac{1}{4}$$

rename $\frac{4}{3}$ as $1 + \frac{1}{3}$

$$= 1 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4}$$

arithmetic

$$= \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4}$$

One thing you should be noticing is that although the ideas being used are simple—adding 0 and multiplying by 1—these ideas can't be fully implemented without appropriate arithmetic skills.

simplifying an expression: a true mathematical sentence results

Remember that to *simplify an expression* means to get a different name for the expression, that in some way is simpler. In the previous example, the author simplified the expression '$\frac{1}{3}$' to get the new name '$\frac14 + \frac13 \cdot \frac14$'. In this situation, the 'new name' appears to be much more complicated than the original name, but is better suited for the current use.

Notice that the process of simplifying $\frac13$ gave rise to a *true mathematical sentence* of the form:

$$\begin{aligned} \textit{original expression} &= \textit{different name} && \textit{(comment)} \\ &= \textit{yet different name} && \textit{(comment)} \\ &= \ \cdots \\ &= \textit{desired name} && \textit{(comment)} \end{aligned}$$

There are two things to notice about this sentence:

- It's a sentence of the form $a = b = c = d = \cdots$; it's just being formatted in a slightly different way:

$$\begin{aligned} a &= b && \textit{(How did we get from } a \textit{ to } b\textit{?)} \\ &= c && \textit{(How did we get from } b \textit{ to } c\textit{?)} \\ &= d && \textit{(How did we get from } c \textit{ to } d\textit{?)} \\ &= \ \cdots \end{aligned}$$

  The different formatting is for aesthetic reasons (to be discussed momentarily), and to easily allow the addition of comments into the sentence.

- The sentence is *true*, because you're just getting different names for the same expression.

$\boxed{\text{Whenever you simplify an expression, a true mathematical sentence of the form } a = b = c = \cdots \text{ results.}}$

Why the special formatting when simplifying an expression?

The format that has been illustrated for simplifying an expression is desirable for the following reasons:

- The original expression stands out at the top of the first (left-most) column.
- The final (desired) name stands out at the bottom of the second column.
- The third (optional) column lets you comment on how you are getting each new name in the simplification process.

'stringing it out' looks cluttered

Notice how cluttered the renaming of $\frac13$ to $\frac14 + \frac13 \cdot \frac14$ looks if the simplification is 'strung out': $\frac13 = \frac13 + (\frac14 - \frac14) = \frac14 + (\frac13 - \frac14) = \frac14 + \frac{1}{12} = \frac14 + \frac{1}{12} \cdot (\frac14 \cdot 4) = \frac14 + (\frac{1}{12} \cdot 4) \cdot \frac14 = \frac14 + \frac13 \cdot \frac14$.

If there aren't too many steps required in simplifying an expression, then you can certainly 'string it out' on one line. But, in general, if you can't fit it on one line, then you should use either the illustrated format, or a slight variation that is appropriate for longer expressions, as discussed next.

SIMPLIFYING AN EXPRESSION
PREFERRED FORMATS

To *simplify an expression* means to get a name for the expression that is, in some way, simpler. The process of *simplifying an expression* results in a mathematical sentence of the form $a = b = c = d = \cdots$ that is *always true*. The preferred formats for simplifying an expression are:

$$\begin{aligned} \textit{original expression} &= \textit{alternate name1} \\ &= \textit{alternate name2} \\ &= \ldots \\ &= \textit{final name} \end{aligned}$$

The original expression stands out in the first column; the final (desired) name stands out at the bottom of the second column. Be sure to line up the '=' signs!

$$\begin{aligned} &\textit{very long original expression} \\ &\qquad = \textit{alternate name1} \\ &\qquad = \ldots \\ &\qquad = \textit{final name} \end{aligned}$$

If the original expression is very long, don't even try to put the first simplification on the same line. Go down to the next line, indent a bit, and start lining up the '=' signs from there.

Comments can optionally be included in either format, as a third column.

If the process of getting from the original expression to the final (desired) name is very short, then you can certainly put it all on one line, like this:

$$\textit{original expression} \quad = \quad \textit{alternate name1} \quad = \quad \ldots \quad = \quad \textit{final name}$$

Regardless of the formatting used, all these sentences give the same information: they tell us that

$$\textit{original expression}\,, \qquad \textit{alternate name1}\,, \qquad \textit{alternate name2}\,, \qquad \ldots\,, \qquad \textit{final name}$$

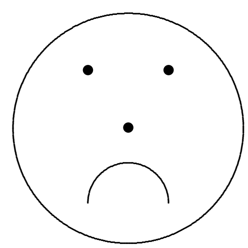
are just different names for the same expression! They can be used interchangeably. The *final name* is the name that is currently desired.

AVOID 'dangling' equal signs

It is considered poor mathematical style to leave an equal sign 'dangling' at the end of a sentence, like this:

$$\begin{aligned} \frac{1}{3} &= \frac{1}{3} + (\frac{1}{4} - \frac{1}{4}) \ = \\ & \frac{1}{4} + (\frac{1}{3} - \frac{1}{4}) \ = \\ & \ldots \end{aligned}$$

DON'T DANGLE!



simplifying an expression versus solving a sentence

One reason so much time has been spent in this text helping you to distinguish between *expressions* and *sentences* is that *you do different things with expressions than you do with sentences*.

The most common thing to do with an *expression*? Simplify it!

The most common thing to do with a *sentence*? Solve it!

The process of 'solving a sentence', and the format to be used when solving a sentence, will be discussed in future sections.

EXERCISES *web practice*	Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

END-OF-SECTION EXERCISES

For problems 13–17: Classify each entry as a mathematical expression (EXP) or a mathematical sentence (SEN).

If an EXPRESSION, then give a simplest name for the expression.

Classify the truth value of each entry that is a sentence: (always) true (T); (always) false (F); or sometimes true/sometimes false (ST/SF).

13. $2 \cdot 8 + 5 \cdot 4$
14. $2 \cdot 8 + 5 \cdot 4 = 16 + 20 = 36$
15. $2 \cdot 8 + 5 \cdot 4 = 16 = 20 = 16 + 20 = 36$
16. $x + 0 = x$
17. $0 = 3 + (-3) = (-5) + 5 = 7 - 7$

SECTION SUMMARY

NUMBERS HAVE LOTS OF DIFFERENT NAMES!

NEW IN THIS SECTION	HOW TO READ	MEANING
special property of 0		For all real numbers x , $x+0=x$. Adding zero to a number doesn't change the number's identity; it doesn't change where the number lives on a number line; it only changes its name.
special property of 1		For all real numbers x , $x \cdot 1 = x$. Multiplying a number by 1 doesn't change the number's identity; it doesn't change where the number lives on a number line; it only changes its name.
theorem	'THEE-rum'	A name that mathematicians give to something that is TRUE and IMPORTANT (★ that has been proved).
proving a result		the process of showing that a result (that is, a theorem, proposition, lemma, corollary, ...) is TRUE
proposition	'prop-a-ZI-shun'	a mathematical result that is not quite important enough to be called a theorem
lemma	'LEM-ma'	a mathematical result that is usually a stepping-stone to a theorem
corollary	'KORE-a-larry'	a mathematical result that is usually an interesting consequence of a theorem
usual names for 0 : For all real numbers x , $x+(-x)=0$		When we get a new name for a number by adding zero, we usually don't use the name 0 for zero. Instead, we use the fact that a number, added to its opposite, always gives 0 .
reciprocal	're-SI-pro-kul'	The reciprocal of a nonzero number x is the new number $\frac{1}{x}$. For example, the reciprocal of 2 is $\frac{1}{2}$.
usual names for 1 : For nonzero real numbers x , $\frac{x}{x} = x \cdot \frac{1}{x} = 1$		When we get a new name for a number by multiplying by 1 , we usually don't use the name 1 for one. Instead, we use the fact that a nonzero number, divided by itself, always gives 1 . Equivalently, a nonzero number, multiplied by its reciprocal, always gives 1 .

<table>
<tr><th>NEW IN THIS SECTION</th><th>HOW TO READ</th><th>MEANING</th></tr>
<tr><td>sentences of the form
$a = b = c$</td><td></td><td>Shorthand for ‘ $a = b$ and $b = c$ ’ .
In order for the sentence ‘ $a = b = c$ ’ to be true, BOTH ‘ $a = b$ ’ and ‘ $b = c$ ’ must be true. When the sentence ‘ $a = b = c$ ’ is true, this means that a , b , and c are just different names for the same number.</td></tr>
<tr><td>What kind of sentence arises when you simplify an expression?</td><td></td><td>Whenever you simplify an expression, a true mathematical sentence of the form $a = b = c = \cdots$ results.</td></tr>
<tr><td>preferred format for simplifying an expression</td><td></td><td>$$\begin{aligned} \textit{original expression} &= \textit{alternate name1} \\ &= \textit{alternate name2} \\ &= \ldots \\ &= \textit{final name} \end{aligned}$$
The original expression stands out in the first column; the final (desired) name stands out at the bottom of the second column. Be sure to line up the ‘=’ signs.</td></tr>
<tr><td>preferred format for simplifying a long expression</td><td></td><td>$$\begin{aligned} &\textit{very long original expression} \\ &\qquad = \textit{alternate name1} \\ &\qquad = \ldots \\ &\qquad = \textit{final name} \end{aligned}$$
If the original expression is very long, don’t even try to put the first simplification on the same line. Go down to the next line, indent a bit, and start lining up the ‘=’ signs from there.</td></tr>
</table>

SOLUTIONS TO EXERCISES: NUMBERS HAVE LOTS OF DIFFERENT NAMES!

IN-SECTION EXERCISES:

1. STEP 1: $\overbrace{0}^{0\text{ pets}} + 2 = 2$; write down the number 2, and circle it.

Since you own fewer than 2 pets, go to STEP 2.

STEP 2: $2 - \overbrace{0}^{0\text{ pets}} = 2$; $2 \cdot 2 = \boxed{4}$; write down the number $\boxed{4}$, and put a box around it. Go to STEP 4.

STEP 4: $\overbrace{0}^{0\text{ pets}} \cdot \overbrace{0}^{0\text{ pets}} = 0$; $0 + \boxed{4} = 4$

2. STEP 1: $\overbrace{7}^{7\text{ pets}} + 2 = 9$; write down the number 9, and circle it.

Since you own more than 2 pets, go to STEP 3.

STEP 3: $\overbrace{7}^{7\text{ pets}} - 2 = 5$; opposite is -5 ; $(-5) \cdot 9 = \boxed{-45}$. Write down the number $\boxed{-45}$ and put a box around it.

STEP 4: $\overbrace{7}^{7\text{ pets}} \cdot \overbrace{7}^{7\text{ pets}} = 49$; $49 + (\boxed{-45}) = 4$

3. $2 \cdot 8 + 5 \cdot 4$: give 2 pieces of candy to each of 8 kids, and 5 pieces of candy to each of 4 kids; OR
give 2 pieces to each of 8 kids, and 4 pieces to each of 5 kids; OR
give 8 pieces to each of 2 kids, and 5 pieces to each of 4 kids.

$5 \cdot 7 + 1$: give 5 pieces of candy to each of 7 kids, with 1 piece left over.

4.

name for 60	information revealed by name
$6 \cdot 10$	give 6 pieces of candy to each of 10 kids; or give 10 pieces of candy to each of 6 kids
$3 \cdot 20$ or $20 \cdot 3$	60 pieces of candy can be evenly distributed among 3 kids, by giving 20 pieces to each
$7 \cdot 8 + 4$ or $8 \cdot 7 + 4$	give 7 pieces of candy to each of 8 kids, with 4 pieces left over
$16 \cdot 3 + 2 \cdot 6$	give 16 pieces to each of 3 kids, and 2 pieces to each of 6 kids; OR give 3 pieces to each of 16 kids, and 6 pieces to each of 2 kids; OR give 16 pieces to each of 3 kids, and 6 pieces to each of 2 kids; OR give 3 pieces to each of 16 kids, and 2 pieces to each of 6 kids.
$\frac{1}{3}(180)$	give one-third of a piece to each of 180 kids

5. The universal set for x is $\mathbb{R}$ because the theorem says 'For all real numbers x ...' .

6. You can add zero to any real number, and this doesn't change the identity of the number. Adding zero gives a new *name* for a number, but doesn't change where it *lives* on a real number line. Consequently, the number 0 is often given the fancy name 'additive identity'.

7. The universal set for x is $\mathbb{R}$ because the theorem says 'For all real numbers x ...' .

8. You can multiply any real number by 1 , and this doesn't change the identity of the number. Multiplying by 1 gives a new *name* for a number, but doesn't change where it *lives* on a real number line. Consequently, the number 1 is often given the fancy name 'multiplicative identity'.

9. (a) $0 = 5 + (-5) = (-5) + 5 = 5 - 5$

(b) $0 = \frac{1}{2} + (-\frac{1}{2}) = (-\frac{1}{2}) + \frac{1}{2} = \frac{1}{2} - \frac{1}{2}$

(c) $0 = 3.2 + (-3.2) = (-3.2) + 3.2 = 3.2 - 3.2$

(d) $0 = (-7) + 7 = 7 + (-7)$

10. (a) $1 = \frac{5}{5} = 5 \cdot \frac{1}{5} = \frac{1}{5} \cdot 5$

(b) $1 = \frac{1/2}{1/2} = \frac{1}{2} \cdot 2 = 2 \cdot \frac{1}{2}$

(c) $1 = \frac{3.2}{3.2} = 3.2 \cdot \frac{1}{3.2} = \frac{1}{3.2} \cdot 3.2$

(d) $1 = \frac{-7}{-7} = -7 \cdot \frac{1}{-7} = \frac{1}{-7} \cdot (-7)$ When a negative number comes after a centered dot, it is customary to put the negative number insides parentheses, because $\frac{1}{-7} \cdot -7$ can look somewhat confusing.

11. (a) true

(b) Since '$2+3 = 5+1$' is false, the entire sentence '$2+3 = 1+5 = 6$' is false. Students sometimes 'string' things together with equal signs as they work through a calculation, using '=' to mean something like 'I'm going on to the next step'. DON'T DO THIS! BE CAREFUL!

(c) true: $1 + 2 + 3 = 1 + 5$ is true, and $1 + 5 = 6$ is true.

(d) true for all nonzero real numbers t; not defined if $t = 0$

(e) true

(f) $1 + (2 + 3) + 4 = 5$ is false; $5 = 1 + 5$ is false; $1 + 5 = 6 + 4$ is false; $6 + 4 = 10$ is true. The entire sentence is FALSE because there is at least one 'piece' that is false. (Indeed, in this case, three of the four subsentences are false!)

12. $a = b = c = d$ is shorthand for: $a = b$ and $b = c$ and $c = d$

END-OF-SECTION EXERCISES:

13. EXP (simplest name: 36)

14. SEN, true

15. SEN, false (Don't use '=' to mean that you're going on to the next step!)

16. SEN, always true

17. SEN, true