16. EXACT VERSUS APPROXIMATE

the numbers get messy

One important use of ‘multiplying by 1 in an appropriate form’ arises in the conversion of units (say, from feet to centimeters), which will be studied soon. Unfortunately, the numbers start getting a bit messy when converting units: numbers like 5280, 2.54, and 1.1 are commonplace. When you need to do arithmetic with numbers like these, it is appropriate to use a calculator. Indeed, most real-life problems involve numbers that are not convenient to work with without calculator assistance. Many calculator-solved problems give an approximate solution, not an exact solution, and the purpose of this section is to increase your awareness of the difference between the two. Here’s an example:

$\frac{1}{7}$ *versus* 0.1428571429

Take your calculator and divide 1 by 7 ; that is, key in the fraction $\frac{1}{7}$. Depending on the current display mode of your calculator, you might see something like 0.1428571429 or 0.1429 . So, are $\frac{1}{7}$ and 0.1428571429 and 0.1429 all just different names for the same number? Well—almost, but not quite. These numbers all lie very close to each other on the number line, but if we were to zoom in, we’d see three different numbers in this order from left to right: $\frac{1}{7}$, 0.1428571429 , and 0.1429 . If $\frac{1}{7}$ is the number that you *really* want, then it’s the exact solution, and the two decimals are approximate solutions.

★

Let $a = \frac{1}{7}$, $b = 0.1428571429$, and $c = 0.1429$. Actually, it would be very difficult to see all three distinct numbers at the same time, because (relatively speaking) a and b are very close together, and c is very far to the right of them. Indeed, the distance between b and c is almost one million times the distance between a and b . Here’s the most accurate representation that can be given with the width restrictions of this page:

In other words, if the number line is magnified so that the distance between a and b is one inch, then the distance between b and c would be almost 16 miles!

equal versus approximately equal

When two numbers x and y live at the same place on the number line, we say ‘ x equals y ’ and write $x = y$.

However, when two numbers x and y are close to each other, but not equal, we say that ‘ x is approximately equal to y ’ and write something like this:

$$x \simeq y$$
$$x \approx y$$
$$x \cong y$$

This text will use the last representation, $x \cong y$.

You should begin to develop an awareness of when numbers are equal, and when they are just approximately equal. This discussion begins with an investigation of the decimal representations of fractions.

simplest form of a fraction

The *simplest form* of a fraction is $\frac{N}{D}$,where N and D have no common factors (except 1). For example, $\frac{6}{15}$ is not in simplest form, because 6 and 15 have a common factor of 3 . Thus,

$$\frac{6}{15} = \frac{3 \cdot 2}{3 \cdot 5} = \frac{3}{3} \cdot \frac{2}{5} = 1 \cdot \frac{2}{5} = \frac{2}{5}\,.$$

Notice that the common factor in the numerator and denominator results in an extra factor of 1 , which can be eliminated, since multiplying by 1 doesn't change a number.

cancelling

The process of getting rid of a common factor in the numerator and denominator is called *cancelling*.

The fraction $\frac{2}{5}$ is now in simplest form, because 2 and 5 have no common factors (except 1).

simplifying a fraction; reducing a fraction; writing a fraction in simplest form

The process of going from $\frac{6}{15}$ to $\frac{2}{5}$ is called *simplifying the fraction*, or *reducing the fraction*, or *writing the fraction in simplest form*.

Here are some common shortcuts for this simplifying process:

$$\frac{6}{15} = \frac{\cancel{3} \cdot 2}{\cancel{3} \cdot 5} = \frac{2}{5}$$

and

(*)

$$\frac{\overset{2}{\cancel{6}}}{\underset{5}{\cancel{15}}} = \frac{2}{5}$$

In (*), the thought process is:

3 goes into 6 twice; cross out the 6 and put 2 ;

3 goes into 15 five times; cross out the 15 and put 5 .

only one level of 'crossing-out', please!

One level of 'crossing-out' is perfectly acceptable, but please resist the temptation to do something like this:

$$\frac{\overset{\overset{2}{\cancel{6}}}{\cancel{12}}}{\underset{\underset{7}{\cancel{21}}}{\cancel{42}}} = \frac{2}{7}$$

Instead, re-write after the first level of crossing-out and continue horizontally:

$$\frac{\overset{6}{\cancel{12}}}{\underset{21}{\cancel{42}}} = \frac{\overset{2}{\cancel{6}}}{\underset{7}{\cancel{21}}} = \frac{2}{7}$$

FACT:
$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$

Here is yet another way that people often write down the process of simplifying a fraction. It is based on the following fact:

$$\begin{aligned} \frac{a}{b} \div \frac{c}{d} &= \frac{a}{b} \cdot \frac{d}{c} \\ &= \frac{a}{b} \cdot \frac{1}{c} \cdot d \\ &= \frac{a}{b} \cdot \frac{\frac{1}{c}}{1} \cdot \frac{1}{\frac{1}{d}} \\ &= \frac{a \cdot \frac{1}{c}}{b \cdot \frac{1}{d}} \\ &= \frac{a \div c}{b \div d} \end{aligned}$$

Thus, the fraction $\frac{a}{b} \div \frac{c}{d}$ also goes by the name $\frac{a \div c}{b \div d}$.

So, if you want to reduce the fraction $\frac{12}{42}$, you can start by finding a number (like 2) that goes into both the numerator and denominator evenly, and then note that dividing by 1 doesn't change a number:

$$\frac{12}{42} = \frac{12}{42} \div \frac{2}{2} = \frac{6}{21} \div \frac{3}{3} = \frac{2}{7}$$

the simplest way to write down the process of simplifying a fraction

Of course, the simplest way to write down the process of simplifying a fraction is to find a number that goes into both the numerator and denominator evenly, do the divisions in your head, and just write down the results:

$$\frac{12}{42} = \frac{6}{21} = \frac{2}{7}$$

summarizing:

four acceptable ways to write the process of simplifying a fraction

Summarizing, here are four acceptable ways to write down the process of simplifying the fraction $\frac{12}{42}$:

$$\frac{12}{42} = \frac{\cancel{2} \cdot 6}{\cancel{2} \cdot 21} = \frac{\cancel{3} \cdot 2}{\cancel{3} \cdot 7} = \frac{2}{7}$$

$$\frac{\overset{6}{\cancel{12}}}{\underset{21}{\cancel{42}}} = \frac{\overset{2}{\cancel{6}}}{\underset{7}{\cancel{21}}} = \frac{2}{7}$$

$$\frac{12}{42} = \frac{12}{42} \div \frac{2}{2} = \frac{6}{21} \div \frac{3}{3} = \frac{2}{7}$$

$$\frac{12}{42} = \frac{6}{21} = \frac{2}{7}$$

You can use whichever format works best for you. However, please be sure to write a complete mathematical sentence that takes you from the starting fraction to the simpler name for the fraction.

EXERCISES

1. Put each fraction in simplest form. Use an acceptable format for writing down the simplifying process.

   a. $\dfrac{3}{12}$

   b. $\dfrac{66}{78}$

   c. $\dfrac{200}{1000}$

   d. $\dfrac{56}{700}$

rational numbers

The *rational numbers* are numbers that can be written in the form $\frac{a}{b}$, where a and b are integers, and b is nonzero. Thus, the **<u>ratio</u>**nal numbers are **<u>ratio</u>**s of integers.

For example, $\frac{2}{5}$ and $\frac{-7}{4}$ are rational numbers.

Also, $5 = \frac{5}{1} = \frac{10}{2} = \ldots$ is a rational number. Indeed, if a number has *one* representation as a ratio of integers, then it has an infinite number of representations.

irrational numbers

Every real number is either rational, or it isn't. If it isn't rational, then it is said to be *irrational*.

a famous irrational number: pi (π)

One of the most famous irrational numbers is π, spelled *pi*, and pronounced like apple 'pie'. Your calculator probably has a key with the symbol π: press it, and you'll get something like this:

$$\pi \cong 3.141592654\,.$$

This decimal is an approximation to π. Indeed, *any* finite decimal is an approximation to π, because the actual decimal name is infinitely long and non-repeating.

π is the ratio of the circumference to the diameter in any circle

The number π gives the ratio of the circumference of a circle (the distance around) to the diameter of the circle (the distance from any point on the circle, through the center, to the opposite side).

Experiment with this! Find any circular object—the bigger the better: a plate, a large bowl, a pot, or the circle provided on this page. Using a flexible ruler, measure the circumference of the circle. Then measure the diameter of the circle. Divide the circumference by the diameter. Depending on the accuracy of your measurements, you'll see digits of π emerge!

EXERCISES

2. Which of the following are rational numbers? If a number is rational, rename it as a ratio of integers.

   a. $\dfrac{1}{5}$

   b. $-\dfrac{25}{13}$

   c. 6

   d. 0

   e. 2π

every rational number is a finite decimal or an infinite repeating decimal

Every rational number can be written as a finite decimal or an infinite repeating decimal.

A finite decimal is one that stops, like 0.157.

An infinite repeating decimal is one that has a specified sequence of digits that repeat, like $0.2637373737\ldots = 0.26\overline{37}$. Notice that in an infinite repeating decimal, the over-bar indicates the digits that repeat.

So, which rational numbers are finite decimals and which are infinite repeating decimals? To answer this question, start by putting the fraction in simplest form, and then factor the denominator into primes.

finite decimal:

only prime factors of 2 and 5 in the denominator

If there are only prime factors of 2 and 5 in the denominator, then the fraction has a finite decimal name. The following example illustrates the idea:

$$\frac{9}{60} = \frac{3}{20} = \frac{3}{2\cdot 2\cdot 5}\cdot\frac{5}{5} = \frac{15}{100} = 0.15$$

If there are only factors of 2 and 5 in the denominator, then additional factors can be introduced, as needed, so that there are equal numbers of 2s and 5s. Then, the denominator is a power of 10, which is easy to write in decimal form.

EXERCISES

3. Write each fraction as a finite decimal, if possible. If this is not possible, so state.
   a. $\dfrac{1}{2\cdot 5\cdot 5\cdot 5}$
   b. $\dfrac{6}{48}$
   c. $\dfrac{2}{3}$
   d. $\dfrac{42}{210}$

infinite repeating decimal:

prime factors other than 2 and 5 in the denominator

When the fraction is in simplest form, then any prime factors other than 2 or 5 in the denominator will give an infinite repeating decimal. Here are some examples.

$$\frac{1}{6} = \frac{1}{2\cdot 3} = 0.166666\ldots = 0.1\overline{6}$$

$$\frac{2}{7} = 0.\overline{285714}$$

$$\frac{3}{11} = 0.\overline{27}$$

Note that the length of the repeating part for the fraction $\frac{2}{7}$ is 6; one less than the denominator (which is 7). Indeed, the length of the repeating part can be at most one less than the denominator. The ★ material next gives the reason why.

★

length of the repeating part for an infinite repeating decimal

To illustrate the idea, consider the fraction $\frac{1}{7}$. Do a long division (see below). When dividing by 7, the only possible remainders are 0 through 6. However, a remainder of 0 would mean the process stops, giving a finite decimal. So, there are only six possible remainders: 1 through 6. As soon as these six remainders are used up, the division scenario repeats itself. Of course, it is possible to get a repeat remainder *before* using up all the possible remainders, which is why the repeating part may not have the maximum possible length.

$$
\begin{array}{rlll}
 & \overline{.142857} & & \\
7\ ) & 1.0 & & \\
 & \underline{7} & & \\
 & \ \ 30 & \longleftarrow & \text{remainder} = 3 \\
 & \ \ \underline{28} & & \\
 & \ \ \ \ 20 & \longleftarrow & \text{remainder} = 2 \\
 & \ \ \ \ \underline{14} & & \\
 & \ \ \ \ \ \ 60 & \longleftarrow & \text{remainder} = 6 \\
 & \ \ \ \ \ \ \underline{56} & & \\
 & \ \ \ \ \ \ \ \ 40 & \longleftarrow & \text{remainder} = 4 \\
 & \ \ \ \ \ \ \ \ \underline{35} & & \\
 & \ \ \ \ \ \ \ \ \ \ 50 & \longleftarrow & \text{remainder} = 5 \\
 & \ \ \ \ \ \ \ \ \ \ \underline{49} & & \\
 & \ \ \ \ \ \ \ \ \ \ \ \ 10 & \longleftarrow & \text{remainder} = 1
\end{array}
$$

next remainder = 3, and process repeats

the other direction

You've just seen that every rational number is either a finite or infinite repeating decimal. It's also true that every finite or infinite repeating decimal is a rational number, as the next couple paragraphs illustrate.

every finite decimal is a rational number

Every finite decimal is a rational number: if there are n decimal places, just multiply by 1 in the form of 10^n over itself:

$$0.1379 = \frac{0.1379}{1} \cdot \frac{10^4}{10^4} = \frac{1379}{10000}\,.$$

Every infinite repeating decimal is also a rational number. The argument uses ideas that come later in the course, so for now it is ★ material.

★

every infinite repeating decimal is a rational number

Here's an example that illustrates the idea.

Let $x = 0.\overline{123}$.

Then, $1000x = 123.\overline{123}$.

Subtracting, $999x = 123$ so $x = \dfrac{123}{999}$.

putting it all together

Now let's put it all together.

There's the set of rational numbers—ratios of integers.

There's the set of all numbers that can be represented as finite decimals or infinite repeating decimals.

The preceding paragraphs have shown that these two sets are identical! Any number in the first is also in the second, and any number in the second is also in the first.

returning to the exact versus approximate idea

If you're working with an infinite repeating decimal, then any finite decimal you write down to represent it is an approximation. *Even if you write down all the digits that your calculator gives you, you still have an approximation.*

So, here are some examples of exact versus approximate:

$$\frac{1}{2} = -0.5$$

$$\frac{2}{7} \cong 0.28571$$

$$\frac{2}{3} \cong 0.66667$$

$$\frac{1}{10^4} = 0.0001$$

EXERCISES

4. Fill in each blank with $=$ or $\cong$:
   a. $\frac{1}{3}$ ______ 0.33333
   b. $\frac{3}{10}$ ______ 0.3
   c. $\frac{1}{2}$ ______ $\frac{3}{6}$
   d. $\frac{1}{13}$ ______ 0.0769230769
   e. 0 ______ $\frac{0}{3}$
   f. -1 ______ $\frac{-5}{5}$

EXERCISES
web practice

Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTIONS TO EXERCISES: EXACT VERSUS APPROXIMATE

1. a. $\frac{3}{12} = \frac{3\cdot 1}{3\cdot 4} = \frac{1}{4}$

b. $\frac{66}{78} = \frac{2\cdot 33}{2\cdot 39} = \frac{3\cdot 11}{3\cdot 13} = \frac{11}{13}$

c. $\frac{200}{1000} = \frac{200\cdot 1}{200\cdot 5} = \frac{1}{5}$

d. $\frac{56}{700} = \frac{7\cdot 8}{7\cdot 100} = \frac{4\cdot 2}{4\cdot 25} = \frac{2}{25}$

2. a, b, c, and d are rational.

a. doesn't need any renaming

b. $-\frac{25}{13} = \frac{-25}{13}$

c. $6 = \frac{6}{1}$

d. $0 = \frac{0}{1}$

e. 2π is not rational (2π is irrational)

3. a. $\frac{1}{2\cdot 5\cdot 5\cdot 5} = \frac{1}{2\cdot 5\cdot 5\cdot 5}\cdot\frac{2\cdot 2}{2\cdot 2} = \frac{4}{10^3} = 0.004$

b. $\frac{6}{48} = \frac{1}{8} = \frac{1}{2\cdot 2\cdot 2}\cdot\frac{5\cdot 5\cdot 5}{5\cdot 5\cdot 5} = \frac{125}{10^3} = 0.125$

c. $\frac{2}{3}$; not possible

d. $\frac{42}{210} = \frac{7\cdot 6}{7\cdot 30} = \frac{6}{30} = \frac{3\cdot 2}{3\cdot 10} = 0.2$

4. a. $\frac{1}{3} \cong 0.33333$

b. $\frac{3}{10} = 0.3$

c. $\frac{1}{2} = \frac{3}{6}$

d. $\frac{1}{13} \cong 0.0769230769$

e. $0 = \frac{0}{3}$

f. $-1 = \frac{-5}{5}$