32. SOLVING LINEAR EQUATIONS IN ONE VARIABLE

classifying families of sentences

In mathematics, it is common to group together sentences of the same type and give them a name. The advantage of classifying families of sentences in this way is that tools can be developed for working with *any* member of the family. Here are some examples:

Sentences like this ...	**have standard form ...**	**and are called ...**
$2x - 5 = \frac{1}{3} - \frac{1}{7}x$	$ax + b = 0\,,\ \ a \ne 0$	linear equations in one variable
$3x^2 - 5x = 1 + 0.3x - x^2$	$ax^2 + bx + c = 0\,,\ a \ne 0$	quadratic equations in one variable
$\sqrt{2}\,x \ge 1 + \frac{2}{9}x$	$ax + b \overbrace{\ <\ }^{\text{or } >, \le, \ge} 0\,,\ \ a \ne 0$	linear inequalities in one variable
$y = 5x + 1$	$ax + by + c = 0$, a and b not both 0	linear equations in two variables

in this section, we'll study linear equations in one variable

In this section we'll practice classifying sentences. Also, we'll study the first type of sentence, called a *linear equation in one variable*. The remaining types of sentences will be studied throughout the rest of the book. So, let's focus attention on the first row of the chart above:

Sentences like this ...	**have standard form ...**	**and are called ...**
$2x - 5 = \frac{1}{3} - \frac{1}{7}x$	$ax + b = 0\,,\ \ a \ne 0$	linear equations in one variable

the standard form tells you what any member of the family looks like

The "standard form" of a sentence makes it easy to talk about a typical member of a family. It gives information about what *any* member of the family looks like—it tells the type of terms that can appear in the sentence. The thought process you should go through when studying a standard form is illustrated next by investigating the standard form "$ax + b = 0\,,\ \ a \ne 0$".

What types of terms are in sentences of the form "$ax + b = 0\,,\ a \ne 0$"?

there must be an x term;

there may or may not be a constant term

When you look at the standard form "$ax + b = 0\,,\ a \ne 0$", here's what you should think:

- This type of sentence is an equation. That is, it has an "$=$" sign.
- There are two types of terms that may appear.

  The term "ax" represents x terms (like $3x$ or $-\frac{2}{5}x$ or $-1.7x$).

  The term "b" represents constant terms (like -4 or $\sqrt{2}$ or $\frac{3}{7}$).
- In the term "ax", "a" is not allowed to equal 0. That is, the number in front of x is not allowed to equal 0. This means that a sentence of the form "$ax + b = 0\,,\ a \ne 0$" *must* have an x term.

  Notice that if $a = 0$, then $ax = 0x = 0$, and there would be no x term.
- The number b does not have any restrictions placed on it, so b is allowed to equal 0. That is, there is *allowed* to be a constant term, but there doesn't have to be one.

Thus, a sentence of the form "$ax + b = 0\,,\ a \ne 0$" *must* have an x term, and may or may not have a constant term.

the variable is usually denoted by x, but might be t or y or ...

The letter x is commonly used as the variable when stating the standard form of a sentence in one variable. However, it is understood that *any* variable could be used. Just keep in mind the convention about using letters near the end of the alphabet to represent real number variables.

If you want to draw attention to the variable being used, you can say the following:

"$3x - 2 = 0$" is a linear equation in x;

"$5t + \frac{1}{3} = 0$" is a linear equation in t;

"$0.4y + \sqrt{2} = 0$" is a linear equation in y; and so on.

it doesn't matter how many terms there are, as long as they're of the correct type

A sentence of the form "$ax + b = 0$, $a \neq 0$" may have *any* number of terms on the left and right sides, as long as they're of the correct type. For example,

$$2x - 3 + x = 5 - 4x - 8x$$

is a sentence of this form. Here's the reason why:

When a mathematician says

a sentence of the form "$ax + b = 0$, $a \neq 0$"

what is *really* meant is

> a sentence that can be *transformed* to the form '$ax+b = 0$, $a \neq 0$" by simplifying expressions and using the Addition and Multiplication Properties of Equality.

Remember that "simplifying an expression" means to rename the expression in a way that is better suited to the current situation. Some common simplifying techniques are combining like terms, using the distributive law, and using properties of fractions (like $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$).

As long as a sentence has only x terms and constant terms, then the sentence can easily be transformed to an equivalent one in the form $ax + b = 0$, $a \neq 0$.

Here's one way that the sentence $2x - 3 + x = 5 - 4x - 8x$ can be transformed:

$2x - 3 + x = 5 - 4x - 8x$	(original equation)
$3x - 3 = 5 - 12x$	(combine like terms on both sides)
$15x - 3 = 5$	(add $12x$ to both sides)
$15x - 8 = 0$	(subtract 5 from both sides)

EXERCISES

1. What tool was used in the previous paragraph, to go from the equation "$3x - 3 = 5 - 12x$" to the equivalent equation "$15x - 3 = 5$"?
2. What tool was used in the previous paragraph, to go from the equation "$15x - 3 = 5$" to the equivalent equation "$15x - 8 = 0$"?

EXAMPLE

deciding if a sentence is of the form $ax+b=0, \ a \ne 0$

Example: Decide if each sentence is of the form $ax+b=0$, $a \ne 0$. That is, decide if each sentence is a linear equation in one variable. If not, give a reason why.

(a) $\frac{1}{3}x - 2 = 0$

Solution: This is a linear equation in one variable. It is already in standard form. It is a linear equation in x.

(b) $\frac{1}{3}x - 5 + x = 0.7x + \sqrt{2}$

Solution: This is a linear equation in one variable. It has only two types of terms, x terms and constant terms. It is a linear equation in x.

(c) $x^2 + x = 5$

Solution: This is **not** a linear equation in one variable. There is not allowed to be an x^2 term.

(d) $3t = 7 + 5t$

Solution: This is a linear equation in one variable. It is a linear equation in t.

(e) $5x - 3 > 0$

Solution: This is **not** a linear equation in one variable. It is not an equation. It is an inequality.

(f) $\dfrac{1}{x} + 3x - 2 = 5$

Solution: This is **not** a linear equation in one variable. There is not allowed to be a $\frac{1}{x}$ term. (The variable x can only appear "upstairs".)

(g) $\dfrac{3x-1}{5} = 4 + \dfrac{1}{7}x$

Solution: This is a linear equation in one variable. If both sides are multiplied by 5 then it is clear that there are only x terms and constant terms. Notice that fractions with only *constants* in the denominator are fine.

(h) $3x - 2t = 5$

Solution: This is **not** a linear equation in one variable. It uses two different variables. It is an equation in two variables.

EXERCISES

3. Decide if each sentence is of the form $ax+b=0$, $a \ne 0$. That is, decide if each sentence is a linear equation in one variable. If not, give a reason why.
   (a) $2 - 5x = 0$
   (b) $\frac{1}{3}t + 5 = -1$
   (c) $0.7 = w - 1.4$
   (d) $x^3 - 3 = 0$
   (e) $\dfrac{1-x}{5} = 2x - 1$
   (f) $2(x - \frac{1}{5}) = 4 + x$
   (g) $\dfrac{2}{x-2} + 3 = -x$
   (h) $2x - 5 > 1 + x$

Here's another example to practice classifying sentences.

EXAMPLE

deciding if a sentence is of the form $ax^2 + bx + c = 0, \ \ a \neq 0$

Example: Decide if each sentence is of the form $ax^2 + bx + c = 0$, $a \neq 0$. A sentence of this form is called a *quadratic equation in one variable*. If not, give a reason why.

Solution: Sentences of this type *must* have an x^2 term; they are *allowed* to have an x term; they are *allowed* to have a constant term. That is, they may or may not have an x term; they may or may not have a constant term.

(a) $3x^2 - \frac{1}{3}x + 2 = 0$

Solution: This is a quadratic equation in one variable; it is in standard form.

(b) $2 - 3x = 5x^2 + \frac{1}{7}x - 1$

Solution: This is a quadratic equation in x.

(c) $8x - 5 = -3x$

Solution: This is **not** a quadratic equation; it has no x^2 term.

(d) $x(x^2 - 1) + 2x^2 + x + 1 = 0$

Solution: This is **not** a quadratic equation; after simplifying, there is an x^3 term, which is not allowed.

(e) $2 - 7t^2 = 4$

Solution: This is a quadratic equation in t.

(f) $y^2 = -3y$

Solution: This is a quadratic equation in y.

EXERCISES

4. Decide if each sentence is of the form $ax^2 + bx + c = 0$, $a \neq 0$. A sentence of this form is called a *quadratic equation in one variable*. If not, give a reason why.

   (a) $1 - 3x = x + 0.2x^2$
   (b) $t^2 - \frac{1}{5} = 0$
   (c) $x^2 - \frac{1}{x} = 0$
   (d) $(x - 1)(x + 3) = 0$
   (e) $x(x - 3) = 1 - 2x^3$
   (f) $\frac{t^2 - 3t}{4} = 1 + 2t$

Here's the precise definition of a linear equation in one variable. The remainder of this section is devoted to solving linear equations in one variable.

DEFINITION

linear equation in one variable

A *linear equation in one variable* is an equation of the form

$$ax + b = 0\,, \quad a \neq 0\,.$$

a linear equation in one variable has exactly one solution

Every linear equation in one variable has exactly one solution. This is easy to see by studying the standard form:

- Every linear equation can be put in the standard form $ax + b = 0$ with $a \neq 0$.
- The following equations are equivalent:

$$\begin{aligned} ax + b &= 0 \\ ax &= -b \\ x &= \frac{-b}{a} = -\frac{b}{a} \end{aligned}$$

The unique solution to $ax + b = 0$ is $x = \frac{-b}{a}$, which can also be written as $x = -\frac{b}{a}$. Indeed, checking gives:

$$\begin{aligned} a(\frac{-b}{a}) + b &\stackrel{?}{=} 0 \\ -b + b &\stackrel{?}{=} 0 \\ 0 &= 0 \end{aligned}$$

It checks!

linear equations in one variable can be solved using basic simplifying techniques and the Addition and Multiplication Properties of Equality

Every linear equation in one variable can be solved using basic simplifying techniques (such as combining like terms and using the distributive law), and the Addition and Multiplication Properties of Equality.

The process is first illustrated by solving the equation $2x - 3 + x = 5 - 4x - 8x$. Notice that when the sentence is in this form, it's certainly *not* easy to see the value of x that makes the sentence true! You would have to think: What number has the property that twice the number, minus three, plus the number, is the same as five minus four times the number minus eight times the number? Who knows? So, the equation is transformed into an equivalent equation that is much easier to work with.

basic procedure for solving a linear equation in one variable

The basic procedure for solving a linear equation in one variable is this:

- First, simplify both sides of the equation as much as possible. In particular, combine like terms.
- Then, get all the x terms on one side (usually the left side).
- Finally, get all the constant terms on the other side (usually the right side).

Write a nice clean list of equivalent equations, ending with one that is so simple that it can be solved by inspection:

$$\begin{aligned} &2x - 3 + x = 5 - 4x - 8x && \text{(the original equation)} \\ &3x - 3 = 5 - 12x && \text{(combine like terms on both sides)} \\ &15x - 3 = 5 && \text{(add } 12x \text{ to both sides)} \\ &15x = 8 && \text{(add 3 to both sides)} \\ &x = \frac{8}{15} && \text{(divide both sides by 15)} \end{aligned}$$

using your calculator to check the solution

You should be able to use your calculator to check the solution, as follows:

- Most calculators have the ability to store a number in memory. You may need to ask your teacher or read the instruction manual to learn how to do this.

  For example, on the TI-83 calculator, you would compute the number, press the $\boxed{\texttt{STO}\triangleright}$ key, press the $\boxed{\texttt{X,T,}\theta\texttt{,n}}$ key, and then press the $\boxed{\texttt{ENTER}}$ key. To recall the stored number, press the $\boxed{\texttt{X,T,}\theta\texttt{,n}}$ key.
- Calculate the number $\frac{8}{15}$, and store it in x.
- Calculate the left side of the original equation, using your stored number x: $2x - 3 + x$. You should get -1.4.
- Calculate the right side of the original equation, using your stored number x: $5 - 4x - 8x$. You should get -1.4.
- If the left and right sides of the equation give the same result, then your solution is correct!

EXAMPLE

solving a linear equation

Here's a second example.

Solve: $3(x-1) - 5x = 2 - 7(2+x)$

Solution:

$$
\begin{aligned}
&3(x-1) - 5x = 2 - 7(2+x) && \text{(the original equation)}\\
&3x - 3 - 5x = 2 - 14 - 7x && \text{(use the distributive law on both sides)}\\
&-2x - 3 = -12 - 7x && \text{(combine like terms on both sides)}\\
&5x - 3 = -12 && \text{(add } 7x \text{ to both sides)}\\
&5x = -9 && \text{(add 3 to both sides)}\\
&x = -\frac{9}{5} && \text{(divide both sides by 5)}
\end{aligned}
$$

Check: When you store $-\frac{9}{5}$ in x, and calculate both sides of the original equation, you should get $0.6 = 0.6$. It checks!

EXERCISES

5. Solve. Write a clean list of equivalent equations. Check your solution.
   (a) $2 - x + 5x = 7x - 3$
   (b) $2(3-x) = -5(x+1) + 4$

solving an equation involving fractions

If an equation involves fractions, just clear the fractions in the first step. To do this, look at all the denominators involved in the fractions, and find a common multiple. (The least common multiple works best if it's easy to find, but any multiple will do.) Then, multiply both sides of the equation by this number, and all the fractions will disappear.

EXAMPLE

solving a linear equation involving fractions

Example: Solve: $2x - 5 = \frac{1}{3} - \frac{1}{7}x$

Notice that 21 is the least common multiple of 3 and 7.

Solution:

$2x - 5 = \frac{1}{3} - \frac{1}{7}x$	(the original equation)
$21(2x-5) = 21(\frac{1}{3} - \frac{1}{7}x)$	(multiply both sides by 21)
$42x - 105 = 7 - 3x$	(multiply out)
$45x - 105 = 7$	(add $3x$ to both sides)
$45x = 112$	(add 105 to both sides)
$x = \frac{112}{45}$	

The fraction $\frac{112}{45}$ is in simplest form (see below for an easy way to use your calculator to check this).

Check: $-0.0\overline{2} = -0.0\overline{2}$. It checks!

putting fractions in simplest form using your calculator

Although you should know the technique for putting fractions in simplest form, it is often more efficient to let your calculator do it for you. You may need to ask your teacher or use your instruction manual to see if your calculator has the capability described next:

To put $\frac{230}{245}$ in simplest form, do the following:

- Divide 230 by 245. You should see something like 0.9387755102.
- Use the "change a decimal to a fraction" capability of your calculator to change 0.9387755102 to a fraction.

  For example, on the TI-83 calculator, press `MATH`, select `▷Frac`, and press `ENTER`.
- The calculator gives you the simplest form, which in this case is $\frac{46}{49}$.

EXERCISES

6. Solve. Write a clean list of equivalent equations. Check your solution.
   (a) $\frac{x}{5} - 2x = 3 + \frac{1}{10}x$
   (b) $\frac{1}{4}(x-2) = \frac{5}{6}x + 1$

solving an equation involving decimals

If an equation involves decimals, just clear the decimals in the first step. To do this, look at all the decimals involved in the equation, and find the most number of decimal places that appear. Multiply by a power of 10 that will get rid of all the decimals.

For example, if only 1 decimal place appears, multiply both sides by 10.

If up to 3 decimal places appear, multiply by 1000.

EXAMPLE

Example: Solve: $x - 0.3 + 0.05x = 2 - 1.4x$

Notice that the decimal 0.05 uses the most number of decimal places (2 decimal places).

Solution:

$x - 0.3 + 0.05x = 2 - 1.4x$	(the original equation)
$100(x - 0.3 + 0.05x) = 100(2 - 1.4x)$	(multiply both sides by 100)
$100x - 30 + 5x = 200 - 140x$	(multiply out)
$105x - 30 = 200 - 140x$	(combine like terms on both sides)
$245x - 30 = 200$	(add $140x$ to both sides)
$245x = 230$	(add 30 to both sides)
$x = \dfrac{230}{245}$	(divide both sides by 145)
$x = \dfrac{46}{49}$	(write the fraction in simplest form)

Check: $0.6857142857 = 0.6857142857$. It checks!

EXERCISES

7. Solve. Write a clean list of equivalent equations. Check your solution.
   (a) $2.61x - 0.003 = x + 1.7$
   (b) $0.4(1 - x) = 0.05(x + 3)$

You don't have to clear fractions and decimals

If you really enjoy doing arithmetic with fractions and decimals, then it is certainly not necessary to clear the fractions and decimals in the first step. As long as you use correct tools in a correct way, you'll always get to the same place!

EXERCISES
web practice

Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTION TO EXERCISES: SOLVING LINEAR EQUATIONS IN ONE VARIABLE

1. the Addition Property of Equality
2. the Addition Property of Equality

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3. (a) $2 - 5x = 0$ is a linear equation in x

(b) $\frac{1}{3}t + 5 = -1$ is a linear equation in t

(c) $0.7 = w - 1.4$ is a linear equation in w

(d) $x^3 - 3 = 0$ is NOT a linear equation; no x^3 term is allowed

(e) $\dfrac{1-x}{5} = 2x - 1$ is a linear equation in x

(f) $2(x - \frac{1}{5}) = 4 + x$ is a linear equation in x

(g) $\dfrac{2}{x-2}+3=-x$ is NOT a linear equation in one variable; the term $\frac{2}{x-2}$ is not allowed

(h) $2x-5>1+x$ is NOT a linear equation; it is not an equation

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4. (a) $1-3x=x+0.2x^2$ is a quadratic equation in x

(b) $t^2-\frac{1}{5}=0$ is a quadratic equation in t

(c) $x^2-\frac{1}{x}=0$ is **not** a quadratic equation; no $\frac{1}{x}$ term is allowed

(d) $(x-1)(x+3)=0$ is a quadratic equation in x (use the distributive law)

(e) $x(x-3)=1-2x^3$ is **not** a quadratic equation; no x^3 term is allowed

(f) $\frac{t^2-3t}{4}=1+2t$ is a quadratic equation in t

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5. (a)

$$\begin{aligned}
&2-x+5x=7x-3 &&\text{(the original equation)}\\
&2+4x=7x-3 &&\text{(combine like terms)}\\
&2-3x=-3 &&\text{(subtract } 7x \text{ from both sides)}\\
&-3x=-5 &&\text{(subtract 2 from both sides)}\\
&x=\frac{-5}{-3}=\frac{5}{3} &&\text{(divide both sides by } -3)
\end{aligned}$$

Check: $8.\overline{6}=8.\overline{6}$. It checks!

(b)

$$\begin{aligned}
&2(3-x)=-5(x+1)+4 &&\text{(the original equation)}\\
&6-2x=-5x-5+4 &&\text{(use the distributive law on both sides)}\\
&6-2x=-5x-1 &&\text{(combine like terms on both sides)}\\
&6+3x=-1 &&\text{(add } 5x \text{ to both sides)}\\
&3x=-7 &&\text{(subtract 6 from both sides)}\\
&x=\frac{-7}{3}=-\frac{7}{3} &&\text{(divide both sides by 3)}
\end{aligned}$$

Check: $10.\overline{6}=10.\overline{6}$. It checks!

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6. (a) Notice that 10 is the least common multiple of 5 and 10.

$$\begin{aligned}
&\frac{x}{5}-2x=3+\frac{1}{10}x &&\text{(the original equation)}\\
&10(\frac{x}{5}-2x)=10(3+\frac{1}{10}x) &&\text{(multiply both sides by 10)}\\
&2x-20x=30+x &&\text{(multiply out)}\\
&-18x=30+x &&\text{(combine like terms)}\\
&-19x=30 &&\text{(subtract } x \text{ from both sides)}\\
&x=\frac{30}{-19}=-\frac{30}{19} &&\text{(divide both sides by } -19)
\end{aligned}$$

Check: $2.842105263=2.842105263$. It checks!

(b) Notice that 12 is the least common multiple of 4 and 6 .

$$\begin{aligned}
&\frac{1}{4}(x-2) = \frac{5}{6}x + 1 && \text{(the original equation)}\\
&12(\frac{1}{4}(x-2)) = 12(\frac{5}{6}x+1) && \text{(multiply both sides by 12)}\\
&3(x-2) = 12(\frac{5}{6}x) + 12 && \text{(simplify)}\\
&3x - 6 = 10x + 12 && \text{(simplify)}\\
&-7x - 6 = 12 && \text{(subtract } 10x \text{ from both sides)}\\
&-7x = 18 && \text{(add 6 to both sides)}\\
&x = \frac{18}{-7} = -\frac{18}{7} && \text{(divide both sides by } -7)
\end{aligned}$$

Check: $-1.142857143 = -1.142857143$. It checks!

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7. (a) Notice that 0.003 uses the most number of decimal places (3 decimal places).

$$\begin{aligned}
&2.61x - 0.003 = x + 1.7 && \text{(the original equation)}\\
&1000(2.61x - 0.003) = 1000(x + 1.7x) && \text{(multiply both sides by 1000)}\\
&2610x - 3 = 1000x + 1700 && \text{(multiply out)}\\
&1610x - 3 = 1700 && \text{(subtract } 1000x \text{ from both sides)}\\
&1610x = 1703 && \text{(add 3 to both sides)}\\
&x = \frac{1703}{1610} && \text{(divide both sides by 1610)}
\end{aligned}$$

Check: $2.757763975 = 2.757763975$. It checks!

(b) Notice that 0.05 uses the most number of decimal places (3 decimal places).

$$\begin{aligned}
&0.4(1-x) = 0.05(x+3) && \text{(the original equation)}\\
&100(0.4(1-x)) = 100(0.05(x+3)) && \text{(multiply both sides by 100)}\\
&40(1-x) = 5(x+3) && \text{(simplify)}\\
&40 - 40x = 5x + 15 && \text{(simplify)}\\
&40 - 45x = 15 && \text{(subtract } 5x \text{ from both sides)}\\
&-45x = -25 && \text{(subtract 40 from both sides)}\\
&x = \frac{-25}{-45} = \frac{5}{9} && \text{(divide both sides by } -45)
\end{aligned}$$

Check: $0.1\overline{7} = 0.1\overline{7}$. It checks!